MATHEMATICAL HANDBOOK

ELEMENTARY MATHEMATICS

M. VYGODSKY

MIR PUBLISHERS • MOSCOW

М. Я. ВЫГОДСКИЙ

СПРАВОЧНИК ПО ЭЛЕМЕНТАРНОЙ МАТЕМАТИКЕ

ИЗДАТЕЛЬСТВО «НАУКА»
МОСКВА

MATHEMATICAL HANDBOOK

ELEMENTARY
MATHEMATICS

M. VYGODSKY

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CONTENTS

Introduction 11

TABLES

1. Some Frequently Encountered Constants 13
2. Squares, Cubes, Square Roots, Cube Roots, Reciprocals, Circumferences, Areas of Circles, Natural Logarithms . . . 14
3. Common Logarithms 18
4. Antilogarithms 23
5. Logarithms of Trigonometric Functions 28
6. Sines and Cosines 36
7. Tangents and Cotangents 40
8. Conversion of Degrees to Radians 48
9. Conversion of Radians to Degrees and Minutes 49
10. Table of Primes < 6000 50
11. Mathematical Symbols 52
12. The Metric System of Measurement 53
13. Some Old Russian Measures 53
14. The Greek Alphabet 53

ARITHMETIC

15. The Subject of Arithmetic 54
16. Whole Numbers (Natural Numbers) 54
17. The Limits of Counting 54
18. The Decimal System of Numeration 55
19. Development of the Number Concept 56
20. Numerals 57
21. Systems of Numeration 57
22. Names of Large Numbers 64
23. Arithmetic Operations 65
24. Order of Operations. Brackets 67
25. Criteria for Divisibility 69
26. Prime and Composite Numbers 71
27. Factorization into Prime Factors 71
28. Greatest Common Divisor 72

29. Least Common Multiple 73
30. Common Fractions 73
31. Reducing Fractions 75
32. Comparing Fractions. Finding a Common Denominator . . 76
33. Adding and Subtracting Fractions 77
34. Multiplication of Fractions. Definition 78
35. Multiplication of Fractions. Rule 79
36. Division of Fractions 80
37. Operations Involving Zero 80
38. The Whole and a Part 82
39. Decimal Fractions 82
40. Properties of Decimal Fractions 84
41. Addition, Subtraction and Multiplication of Decimal Fractions 84
42. Division of a Decimal Fraction by an Integer 85
43. Division Involving Decimal Fractions 87
44. Changing a Decimal Fraction to a Common Fraction and Vice Versa 87
45. Historical Survey of Fractions 89
46. Percentage 90
47. Approximate Calculations 92
48. Notation of Approximate Numbers 93
49. Rules for Rounding Off Numbers 93
50. Absolute and Relative Errors 95
51. Preliminary Rounding Off in Addition and Subtraction 96
52. Error of a Sum and a Difference 98
53. Errors in a Product 100
54. Counting Exact Digits in Multiplication 102
55. Short-Cut Multiplication 104
56. Division of Approximate Numbers 106
57. Short Division 107
58. Involution and Evolution of Approximate Numbers 109
58a. Rule for Extracting a Cube Root 113
59. Mean Quantities 114
60. Abridged Calculation of the Arithmetic Mean 116
61. Accuracy of the Arithmetic Mean 117
62. Ratio and Proportion 118
63. Proportionality (Variation) 119
64. Uses of Proportions. Interpolation 120

ALGEBRA

65. The Subject of Algebra 124
66. Historical Survey of the Development of Algebra 124
67. Negative Numbers 129
68. Negative Numbers (History and Rules of Operation) 131

69. Operations with Negative and Positive Numbers 134
70. Operations with Monomials. Addition and Subtraction of Polynomials 136
71. Multiplication of Sums and Polynomials 138
72. Formulas for Short-Cut Multiplication of Polynomials . . . 138
73. Division of Sums and Polynomials 140
74. Division of a Polynomial by a First-Degree Binomial . . . 142
75. Divisibility of the Binomial $x^m \mp a^m$ by $x \mp a$ 143
76. Factorization of Polynomials 145
77. Algebraic Fractions 146
78. Proportions 147
79. Why We Need Equations 149
80. How to Set Up Equations 150
81. Essential Facts about Equations 151
82. Equivalent Equations. Solving Equations 153
83. Equations Classified 154
84. First-Degree Equation in One Unknown 155
85. A System of Two First-Degree Equations in Two Unknowns 156
86. Solving a System of Two First-Degree Equations in Two Unknowns 158
87. General Formulas for and Special Cases of Solving Systems of Two First-Degree Equations in Two Unknowns 161
88. A System of Three First-Degree Equations in Three Unknowns 163
89. Laws of Exponents 167
90. Operations Involving Radicals 169
91. Irrational Numbers 171
92. Quadratic Equations. Imaginary and Complex Numbers 174
93. Solving a Quadratic Equation 176
94. Properties of the Roots of a Quadratic Equation 179
95. Factoring a Trinomial of the Type ax^2+bx+c 179
96. Higher-Degree Equations Solvable by Means of Quadratics 180
97. A System of Quadratic Equations in Two Unknowns 181
98. Complex Numbers 183
99. Basic Conventions Concerning Complex Numbers 184
100. Addition of Complex Numbers 185
101. Subtraction of Complex Numbers 185
102. Multiplication of Complex Numbers 186
103. Division of Complex Numbers 187
104. Complex Numbers Depicted in the Plane 188
105. The Modulus and Argument of a Complex Number 189
106. Trigonometric Form of a Complex Number 191
107. Geometric Meaning of Addition and Subtraction of Complex Numbers 193
108. Geometric Meaning of Multiplication of Complex Numbers 195
109. Geometric Meaning of Division of Complex Numbers . . . 197

110. Raising a Complex Number to an Integral Power 198
111. Extracting the Root of a Complex Number 200
112. Raising a Complex Number to an Arbitrary Real Power 203
113. Some Facts About Higher Degree Equations 205
114. Fundamentals of Inequalities 207
115. Basic Properties of Inequalities 208
116. Some Important Inequalities 210
117. Equivalent Inequalities. Basic Techniques for Solving Inequalities 213
118. Inequalities Classified 214
119. Inequalities of the First Degree in One Unknown 215
120. A System of Inequalities of the First Degree 215
121. Elementary Inequalities of the Second Degree in One Unknown 216
122. Inequalities of the Second Degree in One Unknown (General Case) 217
123. Arithmetic Progressions 218
124. Geometric Progressions 219
125. Negative, Zero and Fractional Exponents 220
126. The Method of Logarithms. Construction of Logarithmic Tables 223
127. Basic Properties of Logarithms 226
128. Natural Logarithms. The Number *e* 227
129. Common Logarithms 230
130. Operations Involving Artificial Expressions of Negative Logarithms 232
131. Finding the Logarithm of a Number 234
132. Finding a Number from a Logarithm 237
133. Tables of Antilogarithms 239
134. Logarithmic Computations (Worked Examples) 240
135. Combinatorics (Permutations and Combinations) 241
136. The Binomial Theorem 244

GEOMETRY

A. Plane Geometry

137. Geometric Constructions 249
138. The Subject of Geometry 259
139. Historical Survey of the Development of Geometry . . . 259
140. Theorems, Axioms, Definitions 262
141 Straight Line, Ray, Line-Segment 263
142. Angles 263
143. Polygons 265
144. Triangles 266
145. Congruence of Triangles 267
146. Remarkable Lines and Points of the Triangle 268

147. Orthogonal Projections. Relationships Between the Sides of a Triangle 270
148. Parallel Straight Lines 272
149. The Parallelogram and the Trapezoid 274
150. Similarity of Plane Figures Similar Triangles 276
151. Loci and Circles 278
152. Angles in a Circle. The Length of the Circumference and of an Arc 280
152a. Huygens' Formula for Arc Length 282
153. Measuring Angles in a Circle 283
154. Power of a Point 284
155. Radical Axis Radical Centre 286
156. Inscribed and Circumscribed Polygons 288
157. Regular Polygons 289
158. Areas of Plane Figures 291
158a. Approximate Formula for the Area of a Segment 294

B. Solid Geometry

159. General 295
160. Basic Concepts 295
161. Angles 296
162. Projections 298
163. Polyhedral Angles 300
164. Polyhedrons: Prism, Parallelepiped, Pyramid 301
165. Cylinders 305
166. The Cone 306
167. Conic Sections 307
168. The Sphere 308
169. Spherical Polygons 310
170. Parts of a Sphere 312
171. Tangent Plane to a Sphere, a Cylinder and a Cone 314
172. Solid Angles 316
173. Regular Polyhedrons 318
174. Symmetry 319
175. Symmetry of Plane Figures 322
176. Similarity of Solids 323
177. Volumes and Areas of Solids 325

TRIGONOMETRY

178. The Subject of Trigonometry 328
179. Historical Survey of the Development of Trigonometry . . 329
180. Radian Measure of Angles 331

181. Changing from Degrees to Radians and from Radians to Degrees . . . 333
182. Trigonometric Functions of an Acute Angle . . . 334
183. Finding a Trigonometric Function from an Angle . . . 336
184. Finding an Angle from a Trigonometric Function . . . 339
185. Solving Right Triangles . . . 340
186. Table of Logarithms of Trigonometric Functions . . . 341
187. Finding the Logarithm of a Trigonometric Function from the Angle . . . 343
188. Finding the Angle from the Logarithm of the Trigonometric Function . . . 344
189 Solving Right Triangles by Logarithms . . . 346
190. Practical Uses of Right-Triangle Solutions . . . 347
191 Fundamental Relations of Trigonometry . . . 349
192. Trigonometric Functions of an Arbitrary Angle . . . 349
193. Reduction Formulas . . . 352
194. Addition and Subtraction Formulas . . . 355
195. Double-Angle, Triple-Angle and Half-Angle Formulas . . . 355
196. Reducing Trigonometric Expressions to Forms Convenient for Taking Logarithms . . . 356
197. Reducing Expressions Involving the Angles of a Triangle to Logarithmic Form . . . 357
198. Some Important Relations . . . 357
199. Basic Relations Between Elements of a Triangle . . . 359
200. Solving Oblique Triangles . . . 360
201. Inverse Trigonometric Functions (Circular Functions) . . . 365
202. Basic Relations for Inverse Trigonometric Functions . . . 367
203. On the Construction of Tables of Trigonometric Functions 368
204. Trigonometric Equations . . . 370
205. Techniques for Solving Trigonometric Equations . . . 372

FUNCTIONS AND GRAPHS

206. Constants and Variables . . . 377
207. The Functional Relation Between Two Variables . . . 377
208. The Inverse Function . . . 378
209. Representation of a Function by Formula and Table . . . 379
210. Functional Notation . . . 380
211. Coordinates . . . 381
212. Graphical Representation of Functions . . . 382
213. Elementary Functions and Their Graphs . . . 383
214. Graphical Solution of Equations . . . 394
215. Graphical Solution of Inequalities . . . 397
216. Analytical Geometry (Fundamental Notions) . . . 400
217. Limits . . . 402
218. Infinitely Small and Infinitely Large Quantities . . . 403
Index . . . 406

INTRODUCTION

1. Designation of the handbook. The purpose of this book is twofold: firstly, it is designed for quick reference to mathematical concepts (to find out what a tangent is, to compute percentages, to recall formulas for the roots of a quadratic equation, etc.). All definitions, rules, formulas, and theorems are supplied with examples. Where required, hints and suggestions are given as to the use of a rule or how to avoid common mistakes, and so forth.

Secondly, the author believes that this handbook can serve as a manual for reviewing the essentials of mathematics and even as a first introductory course in its practical applications.

2. Handbook and textbook. An attempt has been made to combine the merits of both books in one text. That this has been successful is evident from numerous letters from the readers, most of whom used it as a textbook. True, this handbook differs radically from an ordinary school textbook, where, especially in the senior classes, the emphasis is placed on reasoning: facts are subordinated to logic. This at any rate is how the student regards the process. In this book, the leading role is played by factual material. This does not in the least mean that the reasoning process is absent. Derivations of formulas are given, but only on occasion, as, for example, when it is necessary to stress the central idea of a given section or to overcome any doubts as to the validity of a result (say, when dealing with operations involving complex numbers). In deciding whether to keep a proof or omit it the author was guided by his own teaching experience.

3. How to use the handbook. For quick reference, use the extensive index at the back of the book. If the user has forgotten the exact name of a rule, formula, or mode of solution, he has a detailed table of contents at his disposal in the front.

We strongly advise the user to follow up any additional references he may encounter when investigating a term. Also, much useful information can be gained by reading through the entire section containing the term or concept he is interested in.

It is wise to pay careful attention to the historical surveys contained in each division. They form an integral part of the book and contribute greatly to a deeper understanding of the subject.

The reader who desires to use this manual as a textbook should pay particular attention to the worked examples. Any proofs that are omitted in the handbook can be filled out by reference to a textbook on the subject either at the same time or later. However, it is well to bear in mind that neither handbook nor textbook alone suffices to give the reader a knowledge of the subject: he must use pencil and paper and work through the examples and problems for himself.

TABLES

1. Some Frequently Encountered Constants

Quantity	n	$\log_{10} n$	Quantity	n	$\log_{10} n$
π	3.1416	0.4971	$\sqrt[3]{1:\pi}$	0.6828	$\bar{1}.8343$
2π	6.2832	0.7982	$\sqrt[3]{\pi:6}$	0.8060	$\bar{1}.9063$
3π	9.4248	0.9743	$\sqrt[3]{3:4\pi}$	0.6204	$\bar{1}.7926$
4π	12.5664	1.0992	$\sqrt[3]{\pi^2}$	2.1450	0.3314
$4\pi:3$	4.1888	0.6221	e	2.7183	0.4343
$\pi:2$	1.5708	0.1961	e^2	7.3891	0.8686
$\pi:3$	1.0472	0.0200	$\sqrt{e}$	1.6487	0.2171
$\pi:4$	0.7854	$\bar{1}.8951$	$\sqrt[3]{e}$	1.3956	0.1448
$\pi:6$	0.5236	$\bar{1}.7190$	$1:e$	0.3679	$\bar{1}.5657$
$\pi:180$	0.0175	$\bar{2}.2419$	$1:e^2$	0.1353	$\bar{1}.1314$
$2:\pi$	0.6366	$\bar{1}.8039$	$\sqrt{1:e}$	0.6065	$\bar{1}.7829$
$180:\pi$	57.2958	1.7581	$\sqrt[3]{1:e}$	0.7165	$\bar{1}.8552$
$10800:\pi$	3437.7467	3.5363	$M = \log e$	0.4343	$\bar{1}.6378$
$648000:\pi$	206264.81	5.3144	$\frac{1}{M} = \ln 10$	2.3026	0.3622
$1:\pi$	0.3183	$\bar{1}.5029$	$2!$	2	
$1:2\pi$	0.1592	$\bar{1}.2018$	$3!$	6	
$1:3\pi$	0.1061	$\bar{1}.0257$	$4!$	24	
$1:4\pi$	0.0796	$\bar{2}.9008$	$5!$	120	
π^2	9.8696	0.9943	$6!$	720	
$2\pi^2$	19.7392	1.2953	$7!$	5040	
$\sqrt{\pi}$	1.7725	0.2486	$8!$	40,320	
$\sqrt{2\pi}$	2.5066	0.3991	$9!$	362,880	
$\sqrt{\pi:2}$	1.2533	0.0981	$10!$	3,628,800	
$\sqrt{1:\pi}$	0.5642	$\bar{1}.7514$	$11!$	39,916,800	
$\sqrt{2:\pi}$	0.7979	$\bar{1}.9019$	$12!$	479,001,600	
$\sqrt{3:\pi}$	0.9772	$\bar{1}.9900$			
$\sqrt{4:\pi}$	1.1284	0.0525			
$\sqrt[3]{\pi}$	1.4646	0.1657			

2. Squares, Cubes, Square Roots, Cube Roots, Reciprocals, Circumferences, Areas of Circles, Natural Logarithms

(For three-digit numbers, use interpolation; see Sec. 64. This may result in a slight error in the last digit. On the natural logarithm, ln, see Sec. 128)

n	n^2	n^3	$\sqrt{n}$	$\sqrt{10n}$	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$	$\frac{1}{n}$	πn	$\frac{\pi n^2}{4}$	$\ln n$
1	1	1	1.000	3.162	1.000	2.154	4.642	1.000	3.14	0.785	0.00000
2	4	8	1.414	4.472	1.260	2.714	5.848	0.500	6.28	3.142	0.69315
3	9	27	1.732	5.477	1.442	3.107	6.694	0.333	9.42	7.069	1.09861
4	16	64	2.000	6.325	1.587	3.420	7.368	0.250	12.57	12.566	1.38629
5	25	125	2.236	7.071	1.710	3.684	7.937	0.200	15.71	19.635	1.60944
6	36	216	2.449	7.746	1.817	3.915	8.434	0.167	18.85	28.274	1.79176
7	49	343	2.646	8.367	1.913	4.121	8.879	0.143	21.99	38.484	1.94591
8	64	512	2.828	8.944	2.000	4.309	9.283	0.125	25.13	50.265	2.07944
9	81	729	3.000	9.487	2.080	4.481	9.655	0.111	28.27	63.617	2.19722
10	100	1000	3.162	10.000	2.154	4.642	10.000	0.100	31.42	78.540	2.30259
11	121	1331	3.317	10.488	2.224	4.791	10.323	0.091	34.56	95.033	2.39790
12	144	1728	3.464	10.954	2.289	4.932	10.627	0.083	37.70	113.097	2.48491
13	169	2197	3.606	11.402	2.351	5.066	10.914	0.077	40.84	132.73	2.56495
14	196	2744	3.742	11.832	2.410	5.192	11.187	0.071	43.98	153.94	2.63906
15	225	3375	3.873	12.247	2.466	5.313	11.447	0.067	47.12	176.72	2.70805
16	256	4096	4.000	12.649	2.520	5.429	11.696	0.062	50.27	201.06	2.77259
17	289	4913	4.123	13.038	2.571	5.540	11.935	0.059	53.41	226.98	2.83321
18	324	5832	4.243	13.416	2.621	5.646	12.164	0.056	56.55	254.47	2.89037
19	361	6859	4.359	13.784	2.668	5.749	12.386	0.053	59.69	283.53	2.94444
20	400	8000	4.472	14.142	2.714	5.848	12.599	0.050	62.83	314.16	2.99573

Continued

n	n^2	n^3	$\sqrt{n}$	$\sqrt{10n}$	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$	$\frac{1}{n}$	πn	$\frac{\pi n^2}{4}$	$\ln n$
21	441	9261	4.583	14.491	2.759	5.944	12.806	0.048	65.97	346.36	3.04452
22	484	10,648	4.690	14.832	2.802	6.037	13.006	0.045	69.12	380.13	3.09104
23	529	12,167	4.796	15.166	2.844	6.127	13.200	0.043	72.26	415.48	3.13549
24	576	13,824	4.899	15.492	2.884	6.214	13.389	0.042	75.40	452.39	3.17805
25	625	15,625	5.000	15.811	2.924	6.300	13.572	0.040	78.54	490.87	3.21888
26	676	17,576	5.099	16.125	2.962	6.383	13.751	0.038	81.68	530.93	3.25810
27	729	19,683	5.196	16.432	3.000	6.463	13.925	0.037	84.82	572.55	3.29584
28	784	21,952	5.292	16.733	3.037	6.542	14.095	0.036	87.96	615.75	3.33220
29	841	24,389	5.385	17.029	3.072	6.619	14.260	0.034	91.11	660.52	3.36730
30	900	27,000	5.477	17.321	3.107	6.694	14.422	0.033	94.25	706.86	3.40120
31	961	29,791	5.568	17.607	3.141	6.768	14.581	0.032	97.39	754.77	3.43399
32	1024	32,768	5.657	17.889	3.175	6.840	14.736	0.031	100.53	804.25	3.46574
33	1089	35,937	5.745	18.166	3.208	6.910	14.888	0.030	103.67	855.30	3.49651
34	1156	39,304	5.831	18.439	3.240	6.980	15.037	0.029	106.81	907.92	3.52636
35	1225	42,875	5.916	18.708	3.271	7.047	15.183	0.029	109.96	962.1	3.55535
36	1296	46,656	6.000	18.974	3.302	7.114	15.326	0.028	113.10	1017.9	3.58352
37	1369	50,653	6.083	19.235	3.332	7.179	15.467	0.027	116.24	1075.2	3.61092
38	1444	54,872	6.164	19.494	3.362	7.243	15.605	0.026	119.4	1134.1	3.63759
39	1521	59,319	6.245	19.748	3.391	7.306	15.741	0.026	122.5	1194.6	3.66356
40	1600	64,000	6.325	20.000	3.420	7.368	15.874	0.025	125.7	1256.6	3.68888
41	1681	68,921	6.403	20.248	3.448	7.429	16.005	0.024	128.8	1320.2	3.71357
42	1764	74,088	6.481	20.494	3.476	7.489	16.134	0.024	131.9	1385.4	3.73767
43	1849	79,507	6.557	20.736	3.503	7.548	16.261	0.023	135.1	1452.2	3.76120
44	1936	85,184	6.633	20.976	3.530	7.606	16.386	0.023	138.2	1520.5	3.78419
45	2025	91,125	6.708	21.213	3.557	7.663	16.510	0.022	141.4	1590.4	3.80666

Continued

n	n^2	n^3	$\sqrt{n}$	$\sqrt{10n}$	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$	$\frac{1}{n}$	πn	$\frac{\pi n^2}{4}$	$\ln n$
46	2116	97,336	6.782	21.448	3.583	7.719	16.631	0.022	144.5	1661.9	3.82864
47	2209	103,823	6.856	21.679	3.609	7.775	16.751	0.021	147.7	1734.9	3.85015
48	2304	110,592	6.928	21.909	3.634	7.830	16.869	0.021	150.8	1809.6	3.87120
49	2401	117,649	7.000	22.136	3.659	7.884	16.985	0.020	153.9	1885.7	3.89182
50	2500	125,000	7.071	22.361	3.684	7.937	17.100	0.020	157.1	1963.5	3.91202
51	2601	132,651	7.141	22.583	3.708	7.990	17.213	0.020	160.2	2042.8	3.93183
52	2704	140,608	7.211	22.804	3.733	8.041	17.325	0.019	163.4	2123.7	3.95124
53	2809	148,877	7.280	23.022	3.756	8.093	17.435	0.019	166.4	2206.2	3.97029
54	2916	157,464	7.348	23.238	3.780	8.143	17.544	0.018	169.6	2290.2	3.98898
55	3025	166,375	7.416	23.452	3.803	8.193	17.652	0.018	172.8	2375.8	4.00733
56	3136	175,616	7.483	23.664	3.826	8.243	17.758	0.018	175.9	2463.0	4.02535
57	3249	185,193	7.550	23.875	3.849	8.291	17.863	0.017	179.1	2551.8	4.04306
58	3364	195,112	7.616	24.083	3.871	8.340	17.967	0.017	182.2	2642.1	4.06044
59	3481	205,379	7.681	24.290	3.893	8.387	18.070	0.017	185.4	2734.0	4.07754
60	3600	216,000	7.746	24.495	3.915	8.434	18.171	0.017	188.5	2827.4	4.09434
61	3721	226,981	7.810	24.698	3.936	8.481	18.272	0.016	191.6	2922.5	4.11087
62	3844	238,328	7.874	24.900	3.958	8.527	18.371	0.016	194.8	3019.2	4.12713
63	3969	250,047	7.937	25.100	3.979	8.573	18.469	0.016	197.9	3117.2	4.14313
64	4096	262,144	8.000	25.298	4.000	8.618	18.566	0.016	201.1	3217.0	4.15888
65	4225	274,625	8.062	25.495	4.021	8.662	18.663	0.015	204.2	3318.3	4.17439
66	4356	287,496	8.124	25.690	4.041	8.707	18.758	0.015	207.3	3421.1	4.18965
67	4489	300,763	8.185	25.884	4.062	8.750	18.852	0.015	210.5	3525.6	4.20469
68	4624	314,432	8.246	26.077	4.082	8.794	18.945	0.015	213.6	3631.7	4.21951
69	4761	328,509	8.307	26.268	4.102	8.837	19.038	0.014	216.8	3739.3	4.23411
70	4900	343,000	8.367	26.458	4.121	8.879	19.129	0.014	219.9	3848.4	4.24850
71	5041	357,911	8.426	26.646	4.141	8.921	19.220	0.014	223.1	3959.2	4.26268
72	5184	373,248	8.485	26.833	4.160	8.963	19.310	0.014	226.2	4071.5	4.27667

Continued

n	n^2	n^3	$\sqrt{n}$	$\sqrt{10n}$	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$	$\frac{1}{n}$	πn	$\frac{\pi n^2}{4}$	$\ln n$
73	5329	389,017	8.544	27.019	4.179	9.004	19.399	0.014	229.3	4185.4	4.29046
74	5476	405,224	8.602	27.203	4.198	9.045	19.487	0.013	232.5	4300.8	4.30407
75	5625	421,875	8.660	27.386	4.217	9.086	19.574	0.013	235.6	4417.9	4.31749
76	5776	438,976	8.718	27.568	4.236	9.126	19.661	0.013	238.8	4536.5	4.33073
77	5929	456,533	8.775	27.749	4.254	9.166	19.747	0.013	241.9	4656.6	4.34381
78	6084	474,552	8.832	27.928	4.273	9.205	19.832	0.013	245.0	4778.4	4.35671
79	6241	493,039	8.888	28.107	4.291	9.244	19.916	0.013	248.2	4901.7	4.36945
80	6400	512,000	8.944	28.284	4.309	9.283	20.000	0.012	251.3	5026.6	4.38203
81	6561	531,441	9.000	28.460	4.327	9.322	20.083	0.012	254.5	5153.0	4.39445
82	6724	551,368	9.055	28.636	4.344	9.360	20.165	0.012	257.6	5281.0	4.40672
83	6889	571,787	9.110	28.810	4.362	9.398	20.247	0.012	260.8	5410.6	4.41884
84	7056	592,704	9.165	28.983	4.380	9.435	20.328	0.012	263.9	5541.8	4.43082
85	7225	614,125	9.220	29.155	4.397	9.473	20.408	0.012	267.0	5674.5	4.44265
86	7396	636,056	9.274	29.326	4.414	9.510	20.488	0.012	270.2	5808.8	4.45435
87	7569	658,503	9.327	29.496	4.431	9.546	20.567	0.011	273.3	5944.7	4.46591
88	7744	681,472	9.381	29.665	4.448	9.583	20.646	0.011	276.5	6082.1	4.47734
89	7921	704,969	9.434	29.833	4.465	9.619	20.724	0.011	279.6	6221.1	4.48864
90	8100	729,000	9.487	30.000	4.481	9.655	20.801	0.011	282.7	6361.7	4.49981
91	8281	753,571	9.539	30.166	4.498	9.691	20.878	0.011	285.9	6503.9	4.51086
92	8464	778,688	9.592	30.332	4.514	9.726	20.954	0.011	289.0	6647.6	4.52179
93	8649	804,357	9.644	30.496	4.531	9.761	21.029	0.011	292.2	6792.9	4.53260
94	8836	830,584	9.695	30.659	4.547	9.796	21.105	0.011	295.3	6939.8	4.54329
95	9025	857,375	9.747	30.822	4.563	9.830	21.179	0.011	298.5	7088.2	4.55388
96	9216	884,736	9.798	30.984	4.579	9.865	21.253	0.010	301.6	7238.2	4.56435
97	9409	912,673	9.849	31.145	4.595	9.899	21.327	0.010	304.7	7389.8	4.57471
98	9604	941,192	9.899	31.305	4.610	9.933	21.400	0.010	307.9	7543.0	4.58497
99	9801	970,299	9.950	31.464	4.626	9.967	21.472	0.010	311.0	7697.7	4.59512
00	10,000	1,000,000	10.000	31.623	4.642	10.000	21.544	0.010	314.2	7854.0	4.60517

3. Common Logarithms (see Secs. 131, 132 on how to use the tables: base of natural logarithms: $e = 2.71828$: $\log_{10} e = M = 0.43429$: $\frac{1}{M} = 2.30258$)

Mantissas											Proportional parts		
N	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4 9 13	17 22 26	30 35 39
											4 9 13	17 21 25	30 34 38
											4 8 12	16 21 25	29 33 37
											4 8 12	16 20 24	28 32 36
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 8 12	16 20 24	27 31 35
											4 8 11	15 19 23	27 30 34
											4 7 11	15 18 22	26 29 33
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3 7 11	14 18 21	25 28 32
											3 7 11	14 17 21	24 28 31
											3 7 10	14 17 20	24 27 30
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 7 10	13 17 20	23 27 30
											3 6 10	13 16 19	23 26 29
											3 6 9	13 16 19	22 25 28
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3 6 9	13 16 19	22 25 28
											3 6 9	12 15 18	21 24 27
											3 6 9	11 14 17	20 23 26
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 6 9	11 14 17	20 23 26
											3 5 8	11 14 16	19 22 25

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
16	2041	2068	2095	2122	2143	2175	2201	2227			3	5	8	11	13	16	19	21	24
									2253	2279	3	5	8	10	13	15	18	20	23
17	2304	2330	2355	2380	2405	2430	2455	2480			3	5	8	10	13	15	18	20	23
									2504	2529	2	5	7	10	12	15	17	19	22
18	2553	2577	2601	2625	2648	2672	2695	2718			2	5	7	9	12	14	16	19	21
									2742	2765	2	5	7	9	11	13	16	18	20
19	2788	2810	2833	2856	2878	2900					2	4	7	9	11	14	16	18	20
							2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096						2	4	6	8	11	13	15	17	19
						3118	3139	3160	3181	3201	2	4	6	8	10	12	14	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13

Continued

	Mantissas										Proportional parts								
N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

N	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1 2 2	3 4 5	5 6 7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1 2 2	3 4 5	5 6 7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1 2 2	3 4 5	5 6 7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1 1 2	3 4 4	5 6 7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1 1 2	3 4 4	5 6 7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1 1 2	3 4 4	5 6 6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1 1 2	3 4 4	5 6 6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1 1 2	3 3 4	5 6 6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1 1 2	3 3 4	5 5 6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1 1 2	3 3 4	5 5 6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1 1 2	3 3 4	5 5 6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1 1 2	3 3 4	5 5 6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1 1 2	3 3 4	5 5 6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1 1 2	3 3 4	4 5 6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1 1 2	2 3 4	4 5 6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1 1 2	2 3 4	4 5 6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1 1 2	2 3 4	4 5 5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1 1 2	2 3 4	4 5 5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1 1 2	2 3 4	4 5 5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1 1 2	2 3 4	4 5 5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1 1 2	2 3 3	4 5 5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1 1 2	2 3 3	4 5 5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1 1 2	2 3 3	4 4 5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1 1 2	2 3 3	4 4 5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1 1 2	2 3 3	4 4 5

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4.9.98

Continued

Mantissas											Proportional parts								
N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9150	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

4. Antilogarithms (see Sec. 133 on how to use the tables)

	Numbers										Proportional parts								
m	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3

Continued

	Numbers										Proportional parts								
m	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5

m	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2693	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2927	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	2698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8

Continued

m	Numbers 0	1	2	3	4	5	6	7	8	9	Proportional parts 1	2	3	4	5	6	7	8	9
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4278	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5883	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13

m	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

5. Logarithms of Trigonometric Functions

(see Sec. 186—188; in the column headings log sin, log tan, log cos, all characteristics are increased by 10)

°	′	log sin	d	log tan	cd	log cot	d	log cos		
0	0	−∞	—	−∞	—	+∞		10.0000	0	90
	10	7.4637	3011	7.4637	3011	2.5363		9.999998	50	
	20	7.7648	1760	7.7648	1761	2.2352		9.99999	40	
	30	7.9408	1250	7.9409	1249	2.0591		9.99998	30	
	40	8.0658	969	8.0658	969	1.9342		9.99997	20	
	50	8.1627	792	8.1627	792	1.8373		9.9999	10	
1	0	8.2419	669	8.2419	670	1.7581		9.9999	0	89
	10	8.3088	580	8.3089	580	1.6911		9.9999	50	
	20	8.3668	511	8.3669	512	1.6331		9.9999	40	
	30	8.4179	458	8.4181	457	1.5819	1	9.9999	30	
	40	8.4637	413	8.4638	415	1.5362		9.9998	20	
	50	8.5050	378	8.5053	378	1.4947	1	9.9998	10	
2	0	8.5428	348	8.5431	348	1.4569		9.9997	0	88
	10	8.5776	321	8.5779	322	1.4221	1	9.9997	50	
	20	8.6097	300	8.6101	300	1.3899		9.9996	40	
	30	8.6397	280	8.6401	281	1.3599	1	9.9996	30	
	40	8.6677	263	8.6682	263	1.3318		9.9995	20	
	50	8.6940	248	8.6945	249	1.3055	1	9.9995	10	
3	0	8.7188	235	8.7194	235	1.2806	1	9.9994	0	87
	10	8.7423	222	8.7429	223	1.2571	1	9.9993	50	
	20	8.7645	212	8.7652	213	1.2348	1	9.9993	40	
	30	8.7857	202	8.7865	202	1.2135	1	9.9992	30	
	40	8.8059	192	8.8067	194	1.1933		9.9991	20	
	50	8.8251	185	8.8261	185	1.1739	1	9.9990	10	
4	0	8.8436	177	8.8446	178	1.1554		9.9989	0	86
	10	8.8613	170	8.8624	171	1.1376	1	9.9989	50	
	20	8.8783	163	8.8795	165	1.1205	1	9.9988	40	
	30	8.8946	158	8.8960	158	1.1040	1	9.9987	30	
	40	8.9104	152	8.9118	154	1.0882	1	9.9986	20	
	50	8.9256	147	8.9272	148	1.0728	2	9.9985	10	
		log cos	d	log cot	cd	log tan	d	log sin.	′	°

Continued

°	′	log sin	d	log tan	cd	log cot	d	log cos		
5	0	8.9403	142	8.9420	143	1.0580	1	9.9983	0	85
	10	8.9545	137	8.9563	138	1.0437	1	9.9982	50	
	20	8.9682	134	8.9701	135	1.0299	1	9.9981	40	
	30	8.9816	129	8.9836	130	1.0164	1	9.9980	30	
	40	8.9945	125	8.9966	127	1.0034	2	9.9979	20	
	50	9.0070	122	9.0093	123	0.9907	1	9.9977	10	
6	0	9.0192	119	9.0216	120	0.9784	1	9.9976	0	84
	10	9.0311	115	9.0336	117	0.9664	2	9.9975	50	
	20	9.0426	113	9.0453	114	0.9547	1	9.9973	40	
	30	9.0539	109	9.0467	111	0.9433	1	9.9972	30	
	40	9.0648	107	9.0678	108	0.9322	2	9.9971	20	
	50	9.0755	104	9.0786	105	0.9214	1	9.9969	10	
7	0	9.0859	102	9.0891	104	0.9109	2	9.9968	0	83
	10	9.0961	99	9.0995	101	0.9005	2	9.9966	50	
	20	9.1060	97	9.1096	98	0.8904	1	9.9964	40	
	30	9.1157	95	9.1194	97	0.8806	2	9.9963	30	
	40	9.1252	93	9.1291	94	0.8709	2	9.9961	20	
	50	9.1345	91	9.1385	93	0.8615	1	9.9959	10	
8	0	9.1436	89	9.1478	91	0.8522	2	9.9958	0	82
	10	9.1525	87	9.1569	89	0.8431	2	9.9956	50	
	20	9.1612	85	9.1658	87	0.8342	2	9.9954	40	
	30	9.1697	84	9.1745	86	0.8255	2	9.9952	30	
	40	9.1781	82	9.1831	84	0.8169	2	9.9950	20	
	50	9.1863	80	9.1915	82	0.8085	2	9.9948	10	
9	0	9.1943	79	9.1997	81	0.8003	2	9.9946	0	81
	10	9.2022	78	9.2078	80	0.7922	2	9.9944	50	
	20	9.2100	76	9.2158	78	0.7842	2	9.9942	40	
	30	9.2176	75	9.2236	77	0.7764	2	9.9940	30	
	40	9.2251	73	9.2313	76	0.7687	2	9.9938	20	
	50	9.2324	73	9.2389	74	0.7611	2	9.9936	10	
10	0	9.2397	71	9.2463	73	0.7537	3	9.9934	0	80
	10	9.2468	70	9.2536	73	0.7464	2	9.9931	50	
	20	9.2538	68	9.2609	71	0.7391	2	9.9929	40	
	30	9.2606	68	9.2680	70	0.7320	3	9.9927	30	
	40	9.2674	66	9.2750	69	0.7250	2	9.9924	20	
	50	9.2740	66	9.2819	68	0.7181	3	9.9922	10	
		log cos	d	log cot	cd	log tan	d	log sin		°

Continued

°	′	log sin	d	log tan	cd	log cot	d	log cos		
11	0	9.2806	64	9.2887	66	0.7113	2	9.9919	0	79
	10	9.2870	64	9.2953	67	0.7047	3	9.9917	50	
	20	9.2934	63	9.3020	65	0.6980	2	9.9914	40	
	30	9.2997	61	9.3085	64	0.6915	3	9.9912	30	
	40	9.3058	61	9.3149	63	0.6851	2	9.9909	20	
	50	9.3119	60	9.3212	63	0.6788	3	9.9907	10	
12	0	9.3179	59	9.3275	61	0.6725	3	9.9904	0	78
	10	9.3238	58	9.3336	61	0.6664	2	9.9901	50	
	20	9.3296	57	9.3397	61	0.6603	3	9.9899	40	
	30	9.3353	57	9.3458	59	0.6542	3	9.9896	30	
	40	9.3410	56	9.3517	59	0.6483	3	9.9893	20	
	50	9.3466	55	9.3576	58	0.6424	3	9.9890	10	
13	0	9.3521	54	9.3634	57	0.6366	3	9.9887	0	77
	10	9.3575	54	9.3691	57	0.6309	3	9.9884	50	
	20	9.3629	53	9.3748	56	0.6252	3	9.9881	40	
	30	9.3682	52	9.3804	55	0.6196	3	9.9878	30	
	40	9.3734	52	9.3859	55	0.6141	3	9.9875	20	
	50	9.3786	51	9.3914	54	0.6086	3	9.9872	10	
14	0	9.3837	50	9.3968	53	0.6032	3	9.9869	0	76
	10	9.3887	50	9.4021	53	0.5979	3	9.9866	50	
	20	9.3937	49	9.4074	53	0.5926	4	9.9863	40	
	30	9.3986	49	9.4127	51	0.5873	3	9.9859	30	
	40	9.4035	48	9.4178	52	0.5822	3	9.9856	20	
	50	9.4083	47	9.4230	51	0.5770	4	9.9853	10	
15	0	9.4130	47	9.4281	50	0.5719	3	9.9849	0	75
	10	9.4177	46	9.4331	50	0.5669	3	9.9846	50	
	20	9.4223	46	9.4381	49	0.5619	4	9.9843	40	
	30	9.4269	45	9.4430	49	0.5570	3	9.9839	30	
	40	9.4314	45	9.4479	48	0.5521	4	9.9836	20	
	50	9.4359	44	9.4527	48	0.5473	4	9.9832	10	
16	0	9.4403	44	9.4575	47	0.5425	3	9.9828	0	74
	10	9.4447	44	9.4622	47	0.5378	4	9.9825	50	
	20	9.4491	42	9.4669	47	0.5331	4	9.9821	40	
	30	9.4533	43	9.4716	46	0.5284	3	9.9817	30	
	40	9.4576	42	9.4762	46	0.5238	4	9.9814	20	
	50	9.4618	41	9.4808	45	0.5192	4	9.9810	10	
		log cos	d	log cot	cd	log tan	d	log sin	′	°

Continued

°	′	log sin	d	log tan	cd	log cot	d	log cos		
17	0	9.4659	41	9.4853	45	0.5147	4	9.9806	0	73
	10	9.4700	41	9.4898	45	0.5102	4	9.9802	50	
	20	9.4741	40	9.4943	44	0.5057	4	9.9798	40	
	30	9.4781	40	9.4987	44	0.5013	4	9.9794	30	
	40	9.4821	40	9.5031	44	0.4969	4	9.9790	20	
	50	9.4861	39	9.5075	43	0.4925	4	9.9786	10	
18	0	9.4900	39	9.5118	43	0.4882	4	9.9782	0	72
	10	9.4939	38	9.5161	42	0.4839	4	9.9778	50	
	20	9.4977	38	9.5203	42	0.4797	4	9.9774	40	
	30	9.5015	37	9.5245	42	0.4755	5	9.9770	30	
	40	9.5052	38	9.5287	42	0.4713	4	9.9765	20	
	50	9.5090	36	9.5329	41	0.4671	4	9.9761	10	
19	0	9.5126	37	9.5370	41	0.4630	5	9.9757	0	71
	10	9.5163	36	9.5411	40	0.4589	4	9.9752	50	
	20	9.5199	36	9.5451	40	0.4549	5	9.9748	40	
	30	9.5235	35	9.5491	40	0.4509	4	9.9743	30	
	40	9.5270	36	9.5531	40	0.4469	5	9.9739	20	
	50	9.5306	35	9.5571	40	0.4429	4	9.9734	10	
20	0	9.5341	34	9.5611	39	0.4389	5	9.9730	0	70
	10	9.5375	34	9.5650	39	0.4350	4	9.9725	50	
	20	9.5409	34	9.5689	38	0.4311	5	9.9721	40	
	30	9.5443	34	9.5727	39	0.4273	5	9.9716	30	
	40	9.5477	33	9.5766	38	0.4234	5	9.9711	20	
	50	9.5510	33	9.5804	38	0.4196	4	9.9706	10	
21	0	9.5543	33	9.5842	37	0.4158	5	9.9702	0	69
	10	9.5576	33	9.5879	38	0.4121	5	9.9697	50	
	20	9.5609	32	9.5917	37	0.4083	5	9.9692	40	
	30	9.5641	32	9.5954	37	0.4046	5	9.9687	30	
	40	9.5673	31	9.5991	37	0.4009	5	9.9682	20	
	50	9.5704	32	9.6028	36	0.3972	5	9.9677	10	
22	0	9.5736	31	9.6064	36	0.3936	5	9.9672	0	68
	10	9.5767	31	9.6100	36	0.3900	6	9.9667	50	
	20	9.5798	30	9.6136	36	0.3864	5	9.9661	40	
	30	9.5828	31	9.6172	36	0.3828	5	9.9656	30	
	40	9.5859	30	9.6208	35	0.3792	5	9.9651	20	
	50	9.5889	30	9.6243	36	0.3757	6	9.9646	10	
		log cos	d	log cot	d	log tan	d	log sin	′	°

Continued

°	′	log sin	d	log tan	cd	log cot	d	log cos		
23	0	9.5919	29	9.6279	35	0.3721	5	9.9640	0	67
	10	9.5948	30	9.6314	34	0.3686	6	9.9635	50	
	20	9.5978	29	9.6348	35	0.3652	5	9.9629	40	
	30	9.6007	29	9.6383	34	0.3617	6	9.9624	30	
	40	9.6036	29	9.6417	35	0.3583	5	9.9618	20	
	50	9.6065	28	9.6452	34	0.3548	6	9.9613	10	
24	0	9.6093	28	9.6486	34	0.3514	5	9.9607	0	66
	10	9.6121	28	9.6520	33	0.3480	6	9.9602	50	
	20	9.6149	28	9.6553	34	0.3447	6	9.9596	40	
	30	9.6177	28	9.6587	33	0.3413	6	9.9590	30	
	40	9.6205	27	9.6620	34	0.3380	5	9.9584	20	
	50	9.6232	27	9.6654	33	0.3346	6	9.9579	10	
25	0	9.6259	27	9.6687	33	0.3313	6	9.9573	0	65
	10	9.6286	27	9.6720	32	0.3280	6	9.9567	50	
	20	9.6313	27	9.6752	33	0.3248	6	9.9561	40	
	30	9.6340	26	9.6785	32	0.3215	6	9.9555	30	
	40	9.6366	26	9.6817	33	0.3183	6	9.9549	20	
	50	9.6392	26	9.6850	32	0.3150	6	9.9543	10	
26	0	9.6418	26	9.6882	32	0.3118	7	9.9537	0	64
	10	9.6444	26	9.6914	32	0.3086	6	9.9530	50	
	20	9.6470	25	9.6946	31	0.3054	6	9.9524	40	
	30	9.6495	26	9.6977	32	0.3023	6	9.9518	30	
	40	9.6521	25	9.7009	31	0.2991	7	9.9512	20	
	50	9.6546	24	9.7040	32	0.2960	6	9.9505	10	
27	0	9.6570	25	9.7072	31	0.2928	7	9.9499	0	63
	10	9.6595	25	9.7103	31	0.2897	6	9.9492	50	
	20	9.6620	24	9.7134	31	0.2866	7	9.9486	40	
	30	9.6644	24	9.7165	31	0.2835	6	9.9479	30	
	40	9.6668	24	9.7196	30	0.2804	7	9.9473	20	
	50	9.6692	24	9.7226	31	0.2774	7	9.9466	10	
28	0	9.6716	24	9.7257	30	0.2743	6	9.9459	0	62
	10	9.6740	23	9.7287	30	0.2713	7	9.9453	50	
	20	9.6763	24	9.7317	31	0.2683	7	9.9446	40	
	30	9.6787	23	9.7348	30	0.2652	7	9.9439	30	
	40	9.6810	23	9.7378	30	0.2622	7	9.9432	20	
	50	9.6833	23	9.7408	30	0.2592	7	9.9425	10	
		log cos	d	log cot	cd	log tan	d	log sin	′	°

Continued

°	′	log sin	d	log tan	cd	log cot	d	log cos	′	°
29	0	9.6856	22	9.7438	29	0.2562	7	9.9418	0	61
	10	9.6878	23	9.7467	30	0.2533	7	9.9411	50	
	20	9.6901	22	9.7497	29	0.2503	7	9.9404	40	
	30	9.6923	23	9.7526	30	0.2474	7	9.9397	30	
	40	9.6946	22	9.7556	29	0.2444	7	9.9390	20	
	50	9.6968	22	9.7585	29	0.2415	8	9.9383	10	
30	0	9.6990	22	9.7614	30	0.2386	7	9.9375	0	60
	10	9.7012	21	9.7644	29	0.2356	7	9.9368	50	
	20	9.7033	22	9.7673	28	0.2327	8	9.9361	40	
	30	9.7055	21	9.7701	29	0.2299	7	9.9353	30	
	40	9.7076	21	9.7730	29	0.2270	8	9.9346	20	
	50	9.7097	21	9.7759	29	0.2241	7	9.9338	10	
31	0	9.7118	21	9.7788	28	0.2212	8	9.9331	0	59
	10	9.7139	21	9.7816	29	0.2184	8	9.9323	50	
	20	9.7160	21	9.7845	28	0.2155	7	9.9315	40	
	30	9.7181	20	9.7873	29	0.2127	8	9.9308	30	
	40	9.7201	21	9.7902	28	0.2098	8	9.9300	20	
	50	9.7222	20	9.7930	28	0.2070	8	9.9292	10	
32	0	9.7242	20	9.7958	28	0.2042	8	9.9284	0	58
	10	9.7262	20	9.7986	28	0.2014	8	9.9276	50	
	20	9.7282	20	9.8014	28	0.1986	8	9.9268	40	
	30	9.7302	20	9.8042	28	0.1958	8	9.9260	30	
	40	9.7322	20	9.8070	27	0 1930	8	9.9252	20	
	50	9.7342	19	9.8097	28	0.1903	8	9.9244	10	
33	0	9.7361	19	9.8125	28	0.1875	8	9.9236	0	57
	10	9.7380	20	9.8153	27	0.1847	9	9.9228	50	
	20	9.7400	19	9.8180	28	0.1820	8	9.9219	40	
	30	9.7419	19	9.8208	27	0.1792	8	9.9211	30	
	40	9.7438	19	9.8235	28	0.1765	9	9.9203	20	
	50	9.7457	19	9.8263	27	0.1737	8	9.9194	10	
34	0	9.7476	18	9.8290	27	0.1710	9	9.9186	0	56
	10	9.7494	19	9.8317	27	0.1683	8	9.9177	50	
	20	9.7513	18	9.8344	27	0.1656	9	9.9169	40	
	30	9.7531	19	9.8371	27	0.1629	9	9.9160	30	
	40	9.7550	18	9.8398	27	0.1602	9	9.9151	20	
	50	9.7568	18	9.8425	27	0.1575	8	9.9142	10	
		log cos	d	log cot	cd	log tan	d	log sin	′	°

Continued

°	′	log sin	d	log tan	cd	log cot	d	log cos		
35	0	9.7586	18	9.8452	27	0.1548	9	9.9134	0	55
	10	9.7604	18	9.8479	27	0.1521	9	9.9125	50	
	20	9.7622	18	9.8506	27	0.1494	9	9.9116	40	
	30	9.7640	17	9.8533	26	0.1467	9	9.9107	30	
	40	9.7657	18	9.8559	27	0.1441	9	9.9098	20	
	50	9.7675	17	9.8586	27	0.1414	9	9.9089	10	
36	0	9.7692	18	9.8613	26	0.1387	10	9.9080	0	54
	10	9.7710	17	9.8639	27	0.1361	9	9.9070	50	
	20	9.7727	17	9.8666	26	0.1334	9	9.9061	40	
	30	9.7744	17	9.8692	26	0.1308	10	9.9052	30	
	40	9.7761	17	9.8718	27	0.1282	9	9.9042	20	
	50	9.7778	17	9.8745	26	0.1255	10	9.9033	10	
37	0	9.7795	16	9.8771	26	0.1229	9	9.9023	0	53
	10	9.7811	17	9.8797	27	0.1203	10	9.9014	50	
	20	9.7828	16	9.8824	26	0.1176	9	9.9004	40	
	30	9.7844	17	9.8850	26	0.1150	10	9.8995	30	
	40	9.7861	16	9.8876	26	0.1124	10	9.8985	20	
	50	9 7877	16	9.8902	26	0.1098	10	9.8975	10	
38	0	9.7893	17	9.8928	26	0.1072	11	9.8965	0	52
	10	9.7910	16	9.8954	26	0.1046	10	9.8955	50	
	20	9.7926	15	9.8980	26	0.1020	10	9.8945	40	
	30	9.7941	16	9.9006	26	0.0994	10	9.8935	30	
	40	9.7957	16	9.9032	26	0.0968	10	9.8925	20	
	50	9.7973	16	9.9058	26	0.0942	10	9.8915	10	
39	0	9.7989	15	9.9084	26	0.0916	10	9.8905	0	51
	10	9.8004	16	9.9110	25	0.0890	11	9.8895	50	
	20	9.8020	15	9.9135	26	0.0865	10	9.8884	40	
	30	9.8035	15	9.9161	26	0.0839	10	9.8874	30	
	40	9.8050	16	9.9187	25	0.0813	11	9.8864	20	
	50	9.8066	15	9.9212	26	0.0788	10	9.8853	10	
		log cos	d	log cot	cd	log tan	d	log sin	′	°

Continued

°	′	log sin	d	log tan	cd	log cot	d	log cos		
40	0	9.8081	15	9.9238	26	0.0762	11	9.8843	0	50
	10	9.8096	15	9.9264	25	0.0736	11	9.8832	50	
	20	9.8111	14	9.9289	26	0.0711	11	9.8821	40	
	30	9.8125	15	9.9315	26	0.0685	10	9.8810	30	
	40	9.8140	15	9.9341	25	0.0659	11	9.8800	20	
	50	9.8155	14	9.9366	26	0.0634	11	9.8789	10	
41	0	9.8169	15	9.9392	25	0.0608	11	9.8778	0	49
	10	9.8184	14	9.9417	26	0.0583	11	9.8767	50	
	20	9.8198	15	9.9443	25	0.0557	11	9.8756	40	
	30	9.8213	14	9.9468	26	0.0532	12	9.8745	30	
	40	9.8227	14	9.9494	25	0.0506	11	9.8733	20	
	50	9.8241	14	9.9519	25	0.0481	11	9.8722	10	
42	0	9.8255	14	9.9544	26	0.0456	12	9.8711	0	48
	10	9.8269	14	9.9570	25	0.0430	11	9.8699	50	
	20	9.8283	14	9.9595	26	0.0405	12	9.8688	40	
	30	9.8297	14	9.9621	25	0.0379	11	9.8676	30	
	40	9.8311	13	9.9646	25	0.0354	12	9.8665	20	
	50	9.8324	14	9.9671	26	0.0329	12	9.8653	10	
43	0	9.8338	13	9.9697	25	0.0303	12	9.8641	0	47
	10	9.8351	14	9.9722	25	0.0278	11	9.8629	50	
	20	9.8365	13	9.9747	25	0.0253	12	9.8618	40	
	30	9.8378	13	9.9772	26	0.0228	12	9.8606	30	
	40	9.8391	14	9.9798	25	0.0202	12	9.8594	20	
	50	9.8405	13	9.9823	25	0.0177	13	9.8582	10	
44	0	9.8418	13	9.9848	26	0.0152	12	9.8569	0	46
	10	9.8431	13	9.9874	25	0.0126	12	9.8557	50	
	20	9.8444	13	9.9899	25	0.0101	13	9.8545	40	
	30	9.8457	12	9.9924	25	0.0076	12	9.8532	30	
	40	9.8469	13	9.9949	26	0.0051	13	9.8520	20	
	50	9.8482	13	9.9975	25	0.0025	12	9.8507	10	
45	0	9.8495		10.0000		0.0000		9.8495		45
		log cos	d	log cot	cd	log tan	d	log sin	′	°

6. Sines and Cosines (see Secs. 183, 184 on how to use the tables)

SINES

Deg.	0′ →	10′	20′	30′	40′	50′	60′		Proportional parts								
									1′	2′	3′	4′	5′	6′	7′	8′	9′
↓0	0.0000	0.0029	0.0058	0.0087	0.0116	0.0145	0.0175	89	3	6	9	12	15	17	20	23	26
1	0.0175	0.0204	0.0233	0.0262	0.0291	0.0320	0.0349	88	3	6	9	12	15	17	20	23	26
2	0.0349	0.0378	0.0407	0.0436	0.0465	0.0494	0.0523	87	3	6	9	12	15	17	20	23	26
3	0.0523	0.0552	0.0581	0.0610	0.0640	0.0669	0.0698	86	3	6	9	12	15	17	20	23	26
4	0.0698	0.0727	0.0756	0.0785	0.0814	0.0843	0.0872	85	3	6	9	12	15	17	20	23	26
5	0.0872	0.0901	0.0929	0.0958	0.0987	0.1016	0.1045	84	3	6	9	12	14	17	20	23	26
6	0.1045	0.1074	0.1103	0.1132	0.1161	0.1190	0.1219	83	3	6	9	12	14	17	20	23	26
7	0.1219	0.1248	0.1276	0.1305	0.1334	0.1363	0.1392	82	3	6	9	12	14	17	20	23	26
8	0.1392	0.1421	0.1449	0.1478	0.1507	0.1536	0.1564	81	3	6	9	12	14	17	20	23	26
9	0.1564	0.1593	0.1622	0.1650	0.1679	0.1708	0.1736	80	3	6	9	12	14	17	20	23	26
10	0.1736	0.1765	0.1794	0.1822	0.1851	0.1880	0.1908	79	3	6	9	11	14	17	20	23	26
11	0.1908	0.1937	0.1965	0.1994	0.2022	0.2051	0.2079	78	3	6	9	11	14	17	20	23	26
12	0.2079	0.2108	0.2136	0.2164	0.2193	0.2221	0.2250	77	3	6	9	11	14	17	20	23	26
13	0.2250	0.2278	0.2306	0.2334	0.2363	0.2391	0.2419	76	3	6	8	11	14	17	20	23	25
14	0.2419	0.2447	0.2476	0.2504	0.2532	0.2560	0.2588	75	3	6	8	11	14	17	20	23	25
15	0.2588	0.2616	0.2644	0.2672	0.2700	0.2728	0.2756	74	3	6	8	11	14	17	20	22	25
16	0.2756	0.2784	0.2812	0.2840	0.2868	0.2896	0.2924	73	3	6	8	11	14	17	20	22	25
17	0.2924	0.2952	0.2979	0.3007	0.3035	0.3062	0.3090	72	3	6	8	11	14	17	19	22	25
18	0.3090	0.3118	0.3145	0.3173	0.3201	0.3228	0.3256	71	3	6	8	11	14	17	19	22	25
19	0.3256	0.3283	0.3311	0.3338	0.3365	0.3393	0.3420	70	3	5	8	11	14	16	19	22	25
20	0.3420	0.3448	0.3475	0.3502	0.3529	0.3557	0.3584	69	3	5	8	11	14	16	19	22	25
21	0.3584	0.3611	0.3638	0.3665	0.3692	0.3719	0.3746	68	3	5	8	11	14	16	19	22	24

	60′	50′	40′	30′	20′	10′	0′ ←	Deg.	1′	2′	3′	4′	5′	6′	7′	8′	9′
22	0.3746	0.3773	0.3800	0.3827	0.3854	0.3881	0.3907	67	3	5	8	11	13	16	19	22	24
23	0.3907	0.3934	0.3961	0.3987	0.4014	0.4041	0.4067	66	3	5	8	11	13	16	19	21	24
24	0.4067	0.4094	0.4120	0.4147	0.4173	0.4200	0.4226	65	3	5	8	11	13	16	19	21	24
25	0.4226	0.4253	0.4279	0.4305	0.4331	0.4358	0.4384	64	3	5	8	10	13	16	18	21	24
26	0.4384	0.4410	0.4436	0.4462	0.4488	0.4514	0.4540	63	3	5	8	10	13	16	18	21	23
27	0.4540	0.4566	0.4592	0.4617	0.4643	0.4669	0.4695	62	3	5	8	10	13	15	18	21	23
28	0.4695	0.4720	0.4746	0.4772	0.4797	0.4823	0.4848	61	3	5	8	10	13	15	18	20	23
29	0.4848	0.4874	0.4899	0.4924	0.4950	0.4975	0.5000	60	3	5	8	10	13	15	18	20	23
30	0.5000	0.5025	0.5050	0.5075	0.5100	0.5125	0.5150	59	3	5	8	10	13	15	18	20	23
31	0.5150	0.5175	0.5200	0.5225	0.5250	0.5275	0.5299	58	2	5	7	10	12	15	17	20	22
32	0.5299	0.5324	0.5348	0.5373	0.5398	0.5422	0.5446	57	2	5	7	10	12	15	17	20	22
33	0.5446	0.5471	0.5495	0.5519	0.5544	0.5568	0.5592	56	2	5	7	10	12	15	17	19	22
34	0.5592	0.5616	0.5640	0.5664	0.5688	0.5712	0.5736	55	2	5	7	10	12	14	17	19	22
35	0.5736	0.5760	0.5783	0.5807	0.5831	0.5854	0.5878	54	2	5	7	9	12	14	17	19	21
36	0.5878	0.5901	0.5925	0.5948	0.5972	0.5995	0.6018	53	2	5	7	9	12	14	16	19	21
37	0.6018	0.6041	0.6065	0.6088	0.6111	0.6134	0.6157	52	2	5	7	9	12	14	16	18	21
38	0.6157	0.6180	0.6202	0.6225	0.6248	0.6271	0.6293	51	2	5	7	9	11	14	16	18	20
39	0.6293	0.6316	0.6338	0.6361	0.6383	0.6406	0.6428	50	2	4	7	9	11	13	16	18	20
40	0.6428	0.6450	0.6472	0.6494	0.6517	0.6539	0.6561	49	2	4	7	9	11	13	15	18	20
41	0.6561	0.6583	0.6604	0.6626	0.6648	0.6670	0.6691	48	2	4	7	9	11	13	15	17	20
42	0.6691	0.6713	0.6734	0.6756	0.6777	0.6799	0.6820	47	2	4	6	9	11	13	15	17	19
43	0.6820	0.6841	0.6862	0.6884	0.6905	0.6926	0.6947	46	2	4	6	9	11	13	15	17	19
44	0.6947	0.6967	0.6988	0.7009	0.7030	0.7050	0.7071	↑45	2	4	6	9	10	12	15	17	19

COSINES

SINES *Continued*

Deg.	0′ →	10′	20′	30′	40′	50′	60′		Proportional parts								
									1′	2′	3′	4′	5′	6′	7′	8′	9′
↓45	0.7071	0.7092	0.7112	0.7133	0.7153	0.7173	0.7193	44	2	4	6	8	10	12	14	16	18
46	0.7193	0.7214	0.7234	0.7254	0.7274	0.7294	0.7314	43	2	4	6	8	10	12	14	16	18
47	0.7314	0.7333	0.7353	0.7373	0.7392	0.7412	0.7431	42	2	4	6	8	10	12	14	16	18
48	0.7431	0.7451	0.7470	0.7490	0.7509	0.7528	0.7547	41	2	4	6	8	10	12	14	15	17
49	0.7547	0.7566	0.7585	0.7604	0.7623	0.7642	0.7660	40	2	4	6	8	9	11	13	15	17
50	0.7660	0.7679	0.7698	0.7716	0.7735	0.7753	0.7771	39	2	4	6	7	9	11	13	15	17
51	0.7771	0.7790	0.7808	0.7826	0.7844	0.7862	0.7880	38	2	4	5	7	9	11	13	14	16
52	0.7880	0.7898	0.7916	0.7934	0.7951	0.7969	0.7986	37	2	4	5	7	9	11	12	14	16
53	0.7986	0.8004	0.8021	0.8039	0.8056	0.8073	0.8090	36	2	3	5	7	9	10	12	14	16
54	0.8090	0.8107	0.8124	0.8141	0.8158	0.8175	0.8192	35	2	3	5	7	8	10	12	13	15
55	0.8192	0.8208	0.8225	0.8241	0.8258	0.8274	0.8290	34	2	3	5	7	8	10	12	13	15
56	0.8290	0.8307	0.8323	0.8339	0.8355	0.8371	0.8387	33	2	3	5	6	8	10	11	13	14
57	0.8387	0.8403	0.8418	0.8434	0.8450	0.8465	0.8480	32	2	3	5	6	8	9	11	13	14
58	0.8480	0.8496	0.8511	0.8526	0.8542	0.8557	0.8572	31	2	3	5	6	8	9	11	12	14
59	0.8572	0.8587	0.8601	0.8616	0.8631	0.8646	0.8660	30	1	3	4	6	7	9	10	12	13
60	0.8660	0.8675	0.8689	0.8704	0.8718	0.8732	0.8746	29	1	3	4	6	7	9	10	11	13
61	0.8746	0.8760	0.8774	0.8788	0.8802	0.8816	0.8829	28	1	3	4	6	7	8	10	11	13
62	0.8829	0.8843	0.8857	0.8870	0.0884	0.8897	0.8910	27	1	3	4	5	7	8	9	11	12
63	0.8910	0.8923	0.8936	0.8949	0.8962	0.8975	0.8988	26	1	3	4	5	6	8	9	10	12
64	0.8988	0.9001	0.9013	0.9026	0.9038	0.9051	0.9063	25	1	3	4	5	6	8	9	10	11
65	0.9063	0.9075	0.9088	0.9100	0.9112	0.9124	0.9135	24	1	2	4	5	6	7	8	10	11
66	0.9135	0.9147	0.9159	0.9171	0.9182	0.9194	0.9205	23	1	2	3	5	6	7	8	9	10

	60′	50′	40′	30′	20′	10′	0′ ←	Deg.	1′	2′	3′	4′	5′	6′	7′	8′	9′
67	0.9205	0.9216	0.9228	0.9239	0.9250	0.9261	0.9272	22	1	2	3	4	6	7	8	9	10
68	0.9272	0.9283	0.9293	0.9304	0.9315	0.9325	0.9336	21	1	2	3	4	5	6	7	9	10
69	0.9336	0.9346	0.9356	0.9367	0.9377	0.9387	0.9397	20	1	2	3	4	5	6	7	8	9
70	0.9397	0.9407	0.9417	0.9426	0.9436	0.9446	0.9455	19	1	2	3	4	5	6	7	8	9
71	0.9455	0.9465	0.9474	0.9483	0.9492	0.9502	0.9511	18	1	2	3	4	5	6	6	7	8
72	0.9511	0.9520	0.9528	0.9537	0.9546	0.9555	0.9563	17	1	2	3	3	4	5	6	7	8
73	0.9563	0.9572	0.9580	0.9588	0.9596	0.9605	0.9613	16	1	2	2	3	4	5	6	7	7
74	0.9613	0.9621	0.9628	0.9636	0.9644	0.9652	0.9659	15	1	2	2	3	4	5	5	6	7
75	0.9659	0.9667	0.9674	0.9681	0.9689	0.9696	0.9703	14	1	1	2	3	4	4	5	6	7
76	0.9703	0.9710	0.9717	0.9724	0.9730	0.9737	0.9744	13	1	1	2	3	3	4	5	5	6
77	0.9744	0.9750	0.9757	0.9763	0.9769	0.9775	0.9781	12	1	1	2	3	3	4	4	5	6
78	0.9781	0.9787	0.9793	0.9799	0.9805	0.9811	0.9816	11	1	1	2	2	3	3	4	5	5
79	0.9816	0.9822	0.9827	0.9833	0.9838	0.9843	0.9848	10	1	1	2	2	3	3	4	4	5
80	0.9848	0.9853	0.9858	0.9863	0.9868	0.9872	0.9877	9	0	1	1	2	2	3	3	4	4
81	0.9877	0.9881	0.9886	0.9890	0.9894	0.9899	0.9903	8	0	1	1	2	2	3	3	3	4
82	0.9903	0.9907	0.9911	0.9914	0.9918	0.9922	0.9925	7	0	1	1	2	2	2	3	3	3
83	0.9925	0.9929	0.9932	0.9936	0.9939	0.9942	0.9945	6	0	1	1	1	2	2	2	3	3
84	0.9945	0.9948	0.9951	0.9954	0.9957	0.9959	0.9962	5	0	1	1	1	1	2	2	2	2
85	0.9962	0.9964	0.9967	0.9969	0.9971	0.9974	0.9976	4	0	0	1	1	1	1	2	2	2
86	0.9976	0.9978	0.9980	0.9981	0.9983	0.9985	0.9986	3	0	0	1	1	1	1	1	1	2
87	0.9986	0.9988	0.9989	0.9990	0.9992	0.9993	0.9994	2	0	0	0	1	1	1	1	1	1
88	0.9994	0.9995	0.9996	0.9997	0.9997	0.9998	0.9998	1	0	0	0	0	0	0	1	1	1
89	0.9998	0.9999	0.9999	1.0000	1.0000	1.0000	1.0000	↑0	0	0	0	0	0	0	0	0	0

COSINES

7. Tangents and Cotangents (see Secs. 183, 184)

TANGENTS

Deg.	0′ →	10′	20′	30′	40′	50′	60′		Proportional parts 1′	2′	3′	4′	5′	6′	7′	8′	9′
0↓	0.0000	0.0029	0.0058	0.0087	0.0116	0.0145	0.0175	89	3	6	9	12	15	17	20	23	26
1	0.0175	0.0204	0.0233	0.0262	0.0291	0.0320	0.0349	88	3	6	9	12	15	17	20	23	26
2	0.0349	0.0378	0.0407	0.0437	0.0466	0.0495	0.0524	87	3	6	9	12	15	17	20	23	26
3	0.0524	0.0553	0.0582	0.0612	0.0641	0.0670	0.0699	86	3	6	9	12	15	18	20	23	26
4	0.0699	0.0729	0.0758	0.0787	0.0816	0.0846	0.0875	85	3	6	9	12	15	18	20	23	26
5	0.0875	0.0904	0.0934	0.0963	0.0992	0.1022	0.1051	84	3	6	9	12	15	18	21	23	26
6	0.1051	0.1080	0.1110	0.1139	0.1169	0.1198	0.1228	83	3	6	9	12	15	18	21	24	27
7	0.1228	0.1257	0.1287	0.1317	0.1346	0.1376	0.1405	82	3	6	9	12	15	18	21	24	27
8	0.1405	0.1435	0.1465	0.1495	0.1524	0.1554	0.1584	81	3	6	9	12	15	18	21	24	27
9	0.1584	0.1614	0.1644	0.1673	0.1703	0.1733	0.1763	80	3	6	9	12	15	18	21	24	27
10	0.1763	0.1793	0.1823	0.1853	0.1883	0.1914	0.194	79	3	6	9	12	15	18	21	24	27
11	0.1944	0.1974	0.2004	0.2035	0.2065	0.2095	0.2126	78	3	6	9	12	15	18	21	24	27
12	0.2126	0.2156	0.2186	0.2217	0.2247	0.2278	0.2309	77	3	6	9	12	15	18	21	24	27
13	0.2309	0.2339	0.2370	0.2401	0.2432	0.2462	0.2493	76	3	6	9	12	15	18	22	25	28
14	0.2493	0.2524	0.2555	0.2586	0.2617	0.2648	0.2679	75	3	6	9	12	16	19	22	25	28
15	0.2679	0.2711	0.2742	0.2773	0.2805	0.2836	0.2867	74	3	6	9	13	16	19	22	25	28
16	0.2867	0.2899	0.2931	0.2962	0.2994	0.3026	0.3057	73	3	6	9	13	16	19	22	25	28
17	0.3057	0.3089	0.3121	0.3153	0.3185	0.3217	0.3249	72	3	6	10	13	16	19	22	26	29
18	0.3249	0.3281	0.3314	0.3346	0.3378	0.3411	0.3443	71	3	6	10	13	16	19	23	26	29
19	0.3443	0.3476	0.3508	0.3541	0.3574	0.3607	0.3640	70	3	7	10	13	16	20	23	26	29
20	0.3640	0.3673	0.3706	0.3739	0.3772	0.3805	0.3839	69	3	7	10	13	17	20	23	27	30
21	0.3839	0.3872	0.3906	0.3939	0.3973	0.4006	0.4040	68	3	7	10	13	17	20	24	27	30

	60′	50′	40′	30′	20′	10′	0′ ←	Deg.	1′	2′	3′	4′	5′	6′	7′	8′	9′
22	0.4040	0.4074	0.4108	0.4142	0.4176	0.4210	0.4245	67	3	7	10	14	17	20	24	27	31
23	0.4245	0.4279	0.4314	0.4348	0.4383	0.4417	0.4452	66	3	7	10	14	17	21	24	28	31
24	0.4452	0.4487	0.4522	0.4557	0.4592	0.4628	0.4663	65	4	7	11	14	18	21	25	28	32
25	0.4663	0.4699	0.4734	0.4770	0.4806	0.4841	0.4877	64	4	7	11	14	18	21	25	29	32
26	0.4877	0.4913	0.4950	0.4986	0.5022	0.5059	0.5095	63	4	7	11	15	18	22	25	29	33
27	0.5095	0.5132	0.5169	0,5206	0.5243	0.5280	0.5317	62	4	7	11	15	18	22	26	29	33
28	0.5317	0.5354	0.5392	0.5430	0.5467	0.5505	0.5543	61	4	8	11	15	19	23	26	30	34
29	0.5543	0.5581	0.5619	0.5658	0.5696	0.5735	0.5774	60	4	8	12	15	19	23	27	31	35
30	0.5774	0.5812	0.5851	0.5890	0.5930	0.5969	0.6009	59	4	8	12	16	20	24	27	31	35
31	0.6009	0.6048	0.6088	0.6128	0.6168	0.6208	0.6249	58	4	8	12	16	20	24	28	32	36
32	0.6249	0.6289	0.6330	0.6371	0.6412	0.6453	0.6494	57	4	8	12	16	20	25	29	33	37
33	0.6494	0.6536	0.6577	0.6619	0.6661	0.6703	0.6745	56	4	8	13	17	21	25	29	33	38
34	0.6745	0.6787	0.6830	0.6873	0.6916	0.6959	0.7002	55	4	9	13	17	21	26	30	34	39
35	0.7002	0.7046	0.7089	0.7133	0.7177	0.7221	0.7265	54	4	9	13	18	22	26	31	35	39
36	0.7265	0.7310	0.7355	0.7400	0.7445	0.7490	0.7536	53	5	9	14	18	23	27	32	36	41
37	0.7536	0.7581	0.7627	0.7673	0.7720	0.7766	0.7813	52	5	9	14	19	23	28	32	37	42
38	0.7813	0.7860	0.7907	0.7954	0.8002	0.8050	0.8098	51	5	9	14	19	24	28	33	38	43
39	0.8098	0.8146	0.8195	0.8243	0.8292	0.8342	0.8391	50	5	10	15	20	24	29	34	39	44
40	0.8391	0.8441	0.8491	0.8541	0.8591	0.8642	0.8693	49	5	10	15	20	25	30	35	40	45
41	0.8693	0.8744	0.8796	0.8847	0.8899	0.8952	0.9004	48	5	10	16	21	26	31	36	41	47
42	0.9004	0.9057	0.9110	0.9163	0.9217	0.9271	0.9325	47	5	11	16	21	27	32	38	43	48
43	0.9325	0.9380	0.9435	0.9490	0.9545	0.9601	0.9657	46	6	11	17	22	28	33	39	44	50
44	0.9657	0.9713	0.9770	0.9827	0.9884	0.9942	1.0000	↑45	6	11	17	23	29	34	40	46	51

COTANGENTS

TANGENTS *Continued*

Deg.	0′ →	10′	20′	30′	40′	50′	60′		Proportional parts								
									1′	2′	3′	4′	5′	6′	7′	8′	9′
45↓	1.0000	1.0058	1.0117	1.0176	1.0236	1.0295	1.0355	44	6	12	18	24	30	36	41	47	53
46	1.0355	1.0416	1.0477	1.0538	1.0599	1.0661	1.0724	43	6	12	18	25	31	37	43	49	55
47	1.0724	1.0786	1.0850	1.0913	1.0977	1.1041	1.1106	42	6	13	19	25	32	38	45	51	57
48	1.1106	1.1171	1.1237	1.1303	1.1369	1.1436	1.1504	41	7	13	20	27	33	40	46	53	59
49	1.1504	1.1572	1.1640	1.1708	1.1778	1.1847	1.1918	40	7	14	21	28	34	41	48	55	62
50	1.1918	1.1988	1.2059	1.2131	1.2203	1.2276	1.2349	39	7	14	22	29	36	43	50	57	65
51	1.2349	1.2423	1.2497	1.2572	1.2647	1.2723	1.2799	38	8	15	23	30	38	45	53	60	68
52	1.2799	1.2876	1.2954	1.3032					8	16	23	31	39	47	55	62	70
					1.3111	1.3190	1.3270	37	8	16	24	32	40	48	56	64	72
53	1.3270	1.3351	1.3432	1.3514					8	16	24	32	41	49	57	65	73
					1.3597	1.3680	1.3764	36	8	17	25	33	42	50	58	67	75
54	1.3764	1.3848	1.3937	1.4020					9	17	26	34	43	51	60	68	77
					1.4106	1.4193	1.4282	35	9	17	26	35	44	52	61	70	79
55	1.4282	1.4370	1.4460	1.4550					9	18	27	36	45	54	63	72	81
					1.4641	1.4733	1.4826	34	9	18	28	37	46	55	64	74	83
56	1.4826	1.4919	1.5013	1.5108					9	19	28	38	47	56	66	75	85
					1.5204	1.5301	1.5399	33	10	19	29	39	48	58	68	78	87
57	1.5399	1.5497	1.5597	1.5697					10	20	30	40	50	60	70	80	90
					1.5798	1.5900	1.6003	32	10	20	31	41	51	61	71	82	92
58	1.6003	1.6107	1.6212	1.6318					10	21	31	42	52	63	73	84	94
					1.6426	1.6534	1.6643	31	11	22	32	43	54	65	76	86	97
59	1.6643	1.6753	1.6864	1.6977					11	22	33	44	55	67	78	89	100
					1.7090	1.7205	1.7320	30	11	23	34	46	57	69	80	92	103
60	1.7320	1.7438	1.7556	1.7675					12	24	35	47	59	71	83	94	106
					1.7796	1.7917	1.8040	29	12	24	37	49	61	73	85	98	110

	60′	50′	40′	30′	20′	10′	0′ ←	Deg.	1′	2′	3′	4′	5′	6′	7′	8′	9′
61	1.804	1.816	1.829	1.842	1.855	1.868	1.881	28	1	3	4	5	6	8	9	10	11
62	1.881	1.894	1.907	1.921	1.935	1.949	1.963	27	1	3	4	5	7	8	10	11	11
63	1.963	1.977	1.991	2.006	2.020	2.035	2.050	26	1	3	4	6	7	9	10	12	12
64	2.050	2.066	2.081	2.097	2.112	2.128	2.145	25	2	3	5	6	8	9	11	13	14
65	2.145	2.161	2.177	2.194	2.211	2.229	2.246	24	2	3	5	7	8	10	12	14	15
66	2.246	2.264	2.282	2.300	2.318	2.337	2.356	23	2	4	5	7	9	11	13	15	16
67	2.356	2.375	2.394	2.414	2.434	2.455	2.475	22	2	4	6	8	10	12	14	16	18
68	2.475	2.496	2.517	2.539	2.560	2.583	2.605	21	2	4	6	9	11	13	15	17	19
69	2.605	2.628	2.651	2.675	2.699	2.723	2.747	20	2	5	7	9	12	14	17	19	21
70	2.747	2.773	2.798	2.824	2.850	2.877	2.904	19	3	5	8	10	13	16	18	21	24
71	2.904	2.932	2.960	2.989	3.018	3.047	3.078	18	3	6	9	12	14	17	20	23	26
72	3.078	3.108	3.140	3.172					3	6	9	13	16	19	22	25	28
					3.204	3.237	3.271	17	3	7	10	13	17	20	23	26	30
73	3.271	3.305	3.340	3.376					4	7	11	14	18	21	25	28	31
					3.412	3.450	3.487	16	4	7	11	15	19	22	26	30	33
74	3.487	3.526	3.566	3.606					4	8	12	16	20	24	28	32	36
					3.647	3.689	3.732	15	4	8	13	17	21	25	29	34	38
75	3.732	3.776	3.821	3.867					4	9	13	18	22	27	31	36	40
					3.914	3.962	4.011	14	5	10	14	19	24	29	34	38	43

COTANGENTS

TANGENTS

Continued

A	0′	1′	2′	3′	4′	5′	6′	7′	8′	9′	10′	
76°00′	4.011	4.016	4.021	4.026	4.031	4.036	4.041	4.046	4.051	4.056	4.061	50′
10′	4.061	4.066	4.071	4.076	4.082	4.087	4.092	4.097	4.102	4.107	4.113	40′
20′	4.113	4.118	4.123	4.128	4.134	4.139	4.144	4.149	4.155	4.160	4.165	30′
30′	4.165	4.171	4.176	4.181	4.187	4.192	4.198	4.203	4.208	4.214	4.219	20′
40′	4.219	4.225	4.230	4.236	4.241	4.247	4.252	4.258	4.264	4.269	4.275	10′
50′	4.275	4.280	4.286	4.292	4.297	4.303	4.309	4.314	4.320	4.326	4.331	13°00′
77°00′	4.331	4.337	4.343	4.349	4.355	4.360	4.366	4.372	4.378	4.384	4.390	50′
10′	4.390	4.396	4.402	4.407	4.413	4.419	4.425	4.431	4.437	4.443	4.449	40′
20′	4.449	4.455	4.462	4.468	4.474	4.480	4.486	4.492	4.498	4.505	4.511	30′
30′	4.511	4.517	4.523	4.529	4.536	4.542	4.548	4.555	4.561	4.567	4.574	20′
40′	4.574	4.580	4.586	4.593	4.599	4.606	4.612	4.619	4.625	4.632	4.638	10′
50′	4.638	4.645	4.651	4.658	4.665	4.671	4.678	4.685	4.691	4.698	4.705	12°00′
78′00′	4.705	4.711	4.718	4.725	4.732	4.739	4.745	4.752	4.759	4.766	4.773	50′
10′	4.773	4.780	4.787	4.794	4.801	4.808	4.815	4.822	4.829	4.836	4.843	40′
20′	4.843	4.850	4.857	4.864	4.872	4.879	4.886	4.893	4.901	4.908	4.915	30′
30′	4.915	4.922	4.930	4.937	4.945	4.952	4.959	4.967	4.974	4.982	4.989	20′
40′	4.989	4.997	5.005	5.012	5.020	5.027	5.035	5.043	5.050	5.058	5.066	10′
50′	5.066	5.074	5.081	5.089	5.097	5.105	5.113	5.121	5.129	5.137	5.145	11°00′
79°00′	5.145	5.153	5.161	5.169	5.177	5.185	5.193	5.201	5.209	5.217	5.226	50′
10′	5.226	5.234	5.242	5.250	5.259	5.267	5.276	5.284	5.292	5.301	5.309	40′

	10′	9′	8′	7′	6′	5′	4′	3′	2′	1′	0′	A
20′	5.309	5.318	5.326	5.335	5.343	5.352	5.361	5.369	5.378	5.387	5.396	30′
30′	5.396	5.404	5.413	5.422	5.431	5.440	5.449	5.458	5.466	5.475	5.485	20′
40′	5.485	5.494	5.503	5.512	5.521	5.530	5.539	5.549	5.558	5.567	5.576	10′
50′	5.576	5.586	5.595	5.605	5.614	5.623	5.633	5.642	5.652	5.662	5.671	10°00′
80°00′	5.671	5.681	5.691	5.700	5.710	5.720	5.730	5.740	5.749	5.759	5.769	50′
10′	5.769	5.779	5.789	5.799	5.810	5.820	5.830	5.840	5.850	5.861	5.871	40′
20′	5.871	5.881	5.892	5.902	5.912	5.923	5.933	5.944	5.954	5.965	5.976	30′
30′	5.976	5.986	5.997	6.008	6.019	6.030	6.041	6.051	6.062	6.073	6.084	20′
40′	6.084	6.096	6.107	6.118	6.129	6.140	6.152	6.163	6.174	6.186	6.197	10′
50′	6.197	6.209	6.220	6.232	6.243	6.255	6.267	6.278	6.290	6.302	6.314	9°00′
81°00′	6.314	6.326	6.338	6.350	6.362	6.374	6.386	6.398	6.410	6.423	6.435	50′
10′	6.435	6.447	6.460	6.472	6.485	6.497	6.510	6.522	6.535	6.548	6.561	40′
20′	6.561	6.573	6.586	6.599	6.612	6.625	6.638	6.651	6.665	6.678	6.691	30′
30′	6.691	6.704	6.718	6.731	6.745	6.758	6.772	6.786	6.799	6.813	6.827	20′
40′	6.827	6.841	6.855	6.869	6.883	6.897	6.911	6.925	6.940	6.954	6.968	10′
50′	6.968	6.983	6.997	7.012	7.026	7.041	7.056	7.071	7.085	7.100	7.115	8°00′
82°00′	7.115	7.130	7.146	7.161	7.176	7.191	7.207	7.222	7.238	7.253	7.269	50′
10′	7.269	7.284	7.300	7.316	7.332	7.348	7.364	7.380	7.396	7.412	7.429	40′
20′	7.429	7.445	7.462	7.478	7.495	7.511	7.528	7.545	7.562	7.579	7.596	30′
30′	7.596	7.613	7.630	7.647	7.665	7.682	7.700	7.717	7.735	7.753	7.770	20′
40′	7.770	7.788	7.806	7.824	7.842	7.861	7.879	7.897	7.916	7.934	7.953	10′
50′	7.953	7.972	7.991	8.009	8.028	8.048	8.067	8.086	8.105	8.125	8.144	7°00′

COTANGENTS

TANGENTS *Continued*

A	0′	1′	2′	3′	4′	5′	6′	7′	8′	9′	10′	
83°00′	8.144	8.164	8.184	8.204	8.223	8.243	8.264	8.284	8.304	8.324	8.345	
10′	8.345	8.366	8.386	8.407	8.428	8.449	8.470	8.491	8.513	8.534	8.556	50′
20′	8.556	8.577	8.599	8.621	8.643	8.665	8.687	8.709	8.732	8.754	8.777	40′
30′	8.777	8.800	8.823	8.846	8.869	8.892	8.915	8.939	8.962	8.986	9.010	30′
40′	9.010	9.034	9.058	9.082	9.106	9.131	9.156	9.180	9.205	9.230	9.255	20′
50′	9.255	9.281	9.306	9.332	9.357	9.383	9.409	9.435	9.461	9.488	9.514	10′
												6°00′
84°00′	9.514	9.541	9.568	9.595	9.622	9.649	9.677	9.704	9.732	9.760	9.788	
10′	9.788	9.816	9.845	9.873	9.902	9.931	9.960	9.989	10.02	10.05	10.08	50′
20′	10.08	10.11	10.14	10.17	10.20	10.23	10.26	10.29	10.32	10.35	10.39	40′
30′	10.39	10.42	10.45	10.48	10.51	10.55	10.58	10.61	10.64	10.68	10.71	30′
40′	10.71	10.75	10.78	10.81	10.85	10.88	10.92	10.95	10.99	11.02	11.06	20′
50′	11.06	11.10	11.13	11.17	11.20	11.24	11.28	11.32	11.35	11.39	11.43	10′
												5°00′
85°00′	11.43	11.47	11.51	11.55	11.59	11.62	11.66	11.70	11.74	11.79	11.83	
10′	11.83	11.87	11.91	11.95	11.99	12.03	12.08	12.12	12.16	12.21	12.25	50′
20′	12.25	12.29	12.34	12.38	12.43	12.47	12.52	12.57	12.61	12.66	12.71	40′
30′	12.71	12.75	12.80	12.85	12.90	12.95	13.00	13.05	13.10	13.15	13.20	30′
40′	13.20	13.25	13.30	13.35	13.40	13.46	13.51	13.56	13.62	13.67	13.73	20′
50′	13.73	13.78	13.84	13.89	13.95	14.01	14.07	14.12	14.18	14.24	14.30	10′
												4°00′
86°00′	14.30	14.36	14.42	14.48	14.54	14.61	14.67	14.73	14.80	14.86	14.92	
10′	14.92	14.99	15.06	15.12	15.19	15.26	15.33	15.39	15.46	15.53	15.60	50′
												40′

20′	15.60	15.68	15.75	15.82	15.89	15.97	16.04	16.12	16.20	16.27	16.35	30′
30′	16.35	16.43	16.51	16.59	16.67	16.75	16.83	16.92	17.00	17.08	17.17	20′
40′	17.17	17.26	17.34	17.43	17.52	17.61	17.70	17.79	17.89	17.98	18.07	10′
50′	18.07	18.17	18.27	18.37	18.46	18.56	18.67	18.77	18.87	18.98	19.08	3°00′
87°00′	19.08	19.19	19.30	19.41	19.52	19.63	19.74	19.85	19.97	20.09	20.21	50′
10′	20.21	20.33	20.45	20.57	20.69	20.82	20.95	21.07	21.20	21.34	21.47	40′
20′	21.47	21.61	21.74	21.88	22.02	22.16	22.31	22.45	22.60	22.75	22.90	30′
30′	22.90	23.06	23.21	23.37	23.53	23.69	23.86	24.03	24.20	24.37	24.54	20′
40′	24.54	24.72	24.90	25.08	25.26	25.45	25.64	25.83	26.03	26.23	26.43	10′
50′	26.43	26.64	26.84	27.06	27.27	27.49	27.71	27.94	28.17	28.40	28.64	2°00′
88°00′	28.64	28.88	29.12	29.37	29.62	29.88	30.14	30.41	30.68	30.96	31.24	50′
10′	31.24	31.53	31.82	32.12	32.42	32.73	33.05	33.37	33.69	34.03	34.37	40′
20′	34.37	34.72	35.07	35.43	35.80	36.18	36.56	36.96	37.36	37.77	38.19	30′
30′	38.19	38.62	39.06	39.51	39.97	40.44	40.92	41.41	41.92	42.43	42.96	20′
40′	42.96	43.51	44.07	44.64	45.23	45.83	46.45	47.09	47.74	48.41	49.10	10′
50′	49.10	49.82	50.55	51.30	52.08	52.88	53.71	54.56	55.44	56.35	57.29	1°00′
89°00′	57.29	58.26	59.27	60.31	61.38	62.50	63.66	64.86	66.11	67.40	68.75	50′
10′	68.75	70.15	71.62	73.14	74.73	76.39	78.13	79.94	81.85	83.84	85.94	40′
20′	85.94	88.14	90.46	92.91	95.49	98.22	101.1	104.2	107.4	110.9	114.6	30′
30′	114.6	118.5	122.8	127.3	132.2	137.5	143.2	149.5	156.3	163.7	171.9	20′
40′	171.9	180.9	191.0	202.2	214.9	229.2	245.6	264.4	286.5	312.5	343.8	10′
50′	343.8	382.0	429.7	491.1	573.0	687.5	859.4	1146	1719	3438	∞	0°00′
	10′	9′	8′	7′	6′	5′	4′	3′	2′	1′	0′	*A*

COTANGENTS

8. Conversion of Degrees to Radians (see Sec. 181)

Arc lengths of a circle of radius 1

Deg.	Radians (arc)	Deg.	Radians (arc)	Deg.	Radians (arc)	Min.	Radians (arc)	Min.	Radians (arc)
0	0.0000	35	0.6109	70	1.2217	0	0.0000	30	0.0087
1	0.0175	36	0.6283	71	1.2392	1	0.0003	31	0.0090
2	0.0349	37	0.6458	72	1.2566	2	0.0006	32	0.0093
3	0.0524	38	0.6632	73	1.2741	3	0.0009	33	0.0096
4	0.0698	39	0.6807	74	1.2915	4	0.0012	34	0.0099
5	0.0873	40	0.6981	75	1.3090	5	0.0015	35	0.0102
6	0.1047	41	0.7156	76	1.3265	6	0.0017	36	0.0105
7	0.1222	42	0.7330	77	1.3439	7	0.0020	37	0.0108
8	0.1396	43	0.7505	78	1.3614	8	0.0023	38	0.0111
9	0.1571	44	0.7679	79	1.3788	9	0.0026	39	0.0113
10	0.1745	45	0.7854	80	1.3963	10	0.0029	40	0.0116
11	0.1920	46	0.8029	81	1.4137	11	0.0032	41	0.0119
12	0.2094	47	0.8203	82	1.4312	12	0.0035	42	0.0122
13	0.2269	48	0.8378	83	1.4486	13	0.0038	43	0.0125
14	0.2443	49	0.8552	84	1.4661	14	0.0041	44	0.0128
15	0.2618	50	0.8727	85	1.4835	15	0.0044	45	0.0131
16	0.2793	51	0.8901	86	1.5010	16	0.0047	46	0.0134
17	0.2967	52	0.9076	87	1.5184	17	0.0049	47	0.0137
18	0.3142	53	0.9250	88	1.5359	18	0.0052	48	0.0140
19	0.3316	54	0.9425	89	1.5533	19	0.0055	49	0.0143
20	0.3491	55	0.9599	90	1.5708	20	0.0058	50	0.0145
21	0.3665	56	0.9774	91	1.5882	21	0.0061	51	0.0148
22	0.3840	57	0.9948	92	1.6057	22	0.0064	52	0.0151
23	0.4014	58	1.0123	93	1.6232	23	0.0067	53	0.0154
24	0.4189	59	1.0297	94	1.6406	24	0.0070	54	0.0157
25	0.4363	60	1.0472	95	1.6581	25	0.0073	55	0.0160
26	0.4538	61	1.0647	96	1.6755	26	0.0076	56	0.0163
27	0.4712	62	1.0821	97	1.6930	27	0.0079	57	0.0166
28	0.4887	63	1.0996	98	1.7104	28	0.0081	58	0.0169
29	0.5061	64	1.1170	99	1.7279	29	0.0084	59	0.0172
30	0.5236	65	1.1345	100	1.7453				
31	0.5411	66	1.1519	180	3.1416				
32	0.5585	67	1.1694	200	3.4907				
33	0.5760	68	1.1868	300	5.2360				
34	0.5934	69	1.2043	360	6.2832				

9. Conversion of Radians to Degrees and Minutes (see Sec. 181)

Radians	Deg. and min.	Radians	Deg. and min.	Radians	Deg. and min.	Radians	Min.	Radians	Min.
1	57°18′	0.1	5°44′	0.01	0°34′	0.001	0°03′	0.0001	0°00′
2	114°35′	0.2	11°28′	0.02	1°09′	0.002	0°07′	0.0002	0°01′
3	171°53′	0.3	17°11′	0.03	1°43′	0.003	0°10′	0.0003	0°01′
4	229°11′	0.4	22°55′	0.04	2°18′	0.004	0°14′	0.0004	0°01′
5	286°29′	0.5	28°39′	0.05	2°52′	0.005	0°17′	0.0005	0°02′
6	343°46′	0.6	34°23′	0.06	3°26′	0.006	0°21′	0.0006	0°02′
7	401°04′	0.7	40°06′	0.07	4°01′	0.007	0°24′	0.0007	0°02′
8	458°22′	0.8	45°50′	0.08	4°35′	0.008	0°28′	0.0008	0°03′
9	515°40′	0.9	51°34′	0.09	5°09′	0.009	0°31′	0.0009	0°03′

10. Table of Primes < 6000

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11. Mathematical Symbols

Symbol	*Meaning*	*Example*
$=$	is equal to	$a=b$
$\neq$	is not equal to	$a\neq b$
$\approx$	is approximately equal to	$a\approx b$
$>$	is greater than	$5>2$
$<$	is less than	$3<10$
$\geqslant$	is greater than or equal to	$a\geqslant b$
$\leqslant$	is less than or equal to	$a\leqslant b$
$\lvert\ \rvert$	absolute value	$\lvert a\rvert$
$\sqrt[n]{\ }$	*n*th root of (usually means the principal *n*th root)	$\sqrt[3]{8}=2$
$!$	factorial	$5!=1\cdot 2\cdot 3\cdot 4\cdot 5=120$ (read: factorial 5, or 5 factorial)
$\log_b$	logarithm to the base b	$\log_2 8=3$
$\log_{10}$ (or log)	common logarithm ($\log a$ is used for $\log_{10} a$ when the context shows that the base is 10)	$\log 100=2$
ln (or $\log_e$)	natural (Napierian) logarithm (to the base e)	
lim	limit	
const	constant	
Σ	summation of	
$\triangle$	triangle	$\triangle ABC$ (pl. $\triangle$s ABC and DEF)
$\angle$	angle	$\angle ABC$
$\smile$	arc	$\overset{\smile}{AB}$
$\parallel$	is parallel to	$AB\parallel CD$
$\perp$	is perpendicular to	$AB\perp CD$
$\sim$	is similar to	$\triangle ABC\sim\triangle DEF$
π	ratio of the circumference of a circle to the diameter, equal to 3.1415926536 ...	
$^\circ$	degree	$10^\circ\,30'\,35''$
$'$	minute	
$''$	second	
sin	sine	$\sin 30^\circ=\frac{1}{2}$
cos	cosine	$\cos\frac{\pi}{2}=0$
tan	tangent	$\tan 40^\circ=0.8391$
cot	cotangent	$\cot 25^\circ 10'=2.128$
sec	secant	$\sec 60^\circ=2$
csc	cosecant	$\csc 90^\circ=1$
arcsin	arc sine	$\arcsin\frac{1}{2}=30^\circ$
arccos	arc cosine	$\arccos 0=\frac{\pi}{2}$
arctan	arc tangent	$\arctan 0.8391=40^\circ$
arccot	arc cotangent	$\operatorname{arccot} 2.128=25^\circ 10'$
arcsec	arc secant	$\operatorname{arcsec} 2=60^\circ$
arccsc	arc cosecant	$\operatorname{arccsc} 1=90^\circ$

12. The Metric System of Measurement

Linear Measure

1 kilometre (km) = 1000 metres (m)
1 metre (m) = 10 decimetres (dm) = 100 centimetres (cm)
1 decimetre (dm) = 10 centimetres (cm)
1 centimetre (cm) = 10 millimetres (mm)

Square Measure (area)

1 square kilometre (km^2) = 1,000,000 square metres (m^2)
1 square metre (m^2) = 100 square decimetres (dm^2) = 10,000 square centimetres (cm^2)
1 hectare (ha) = 100 ares = 10,000 square metres (m^2)
1 are = 100 square metres (m^2)

Cubic Measure (volume)

1 cubic metre (m^3) = 1000 cubic decimetres (dm^3) = 1,000,000 cubic centimetres (cm^3)
1 cubic decimetre (dm^3) = 1000 cubic centimetres (cm^3)
1 litre (l) = 1 cubic decimetre (dm^3)
1 hectolitre (hl) = 100 litres (l)

Metric Weight

1 ton (metric) = 1000 kilograms (kg)
1 centner = 100 kilograms (kg)
1 kilogram (kg) = 1000 grams (g)
1 gram (g) = 1000 milligrams (mg)

13. Some Old Russian Measures

Linear Measure

1 versta = 500 sagenes = 1500 arshins = 3500 feet = 1066.8 m
1 sagene = 3 arshins = 48 verchoks = 7 feet = 84 inches = 2.1336 m
1 arshin = 16 verchoks = 71.12 cm
1 verchok = 4.450 cm
1 foot = 12 inches = 0.3048 m
1 inch = 2.540 cm
1 nautical mile = 1852.2 m in USSR, 1853.18 m in Britain, 1853.25 m in USA

Weight

1 pood = 40 pounds = 16.380 kg
1 pound = 0.40951 kg

14. The Greek Alphabet

Αα	alpha	Νν	nu
Ββ	beta	Ξξ	xi
Γγ	gamma	Οο	omicron
Δδ	delta	Ππ	pi
Εε	epsilon	Ρρ	rho
Ζζ	zeta	Σσ	sigma
Ηη	eta	Ττ	tau
Θθ	theta	Υυ	upsilon
Ιι	iota	Φφ	phi
Κϰ	kappa	Χχ	chi
Λλ	lambda	Ψψ	psi
Μμ	mu	Ωω	omega

ARITHMETIC

15. The Subject of Arithmetic

Arithmetic is the science of numbers, the name stemming from the Greek word "arithmos" which means "number". It involves the most elementary properties of numbers and rules of calculation. Deeper properties of numbers are studied in the theory of numbers.

16. Whole Numbers (Natural Numbers)

The first conceptions of number were acquired by man in remote antiquity (see Sec. 17). It began with the counting of people, animals, and the various articles and possessions of primitive man. Counting produced the numbers one, two, three, etc., which are now called *natural numbers*. In arithmetic they are also referred to as *whole numbers* or integers (the term "integer" has a broader meaning in mathematics; see Sec. 67).

The concept of a natural number is one of the most elementary notions. The only way to explain it is by demonstration. In the third century B. C., Euclid defined number (natural number) as a "collection made up of units"; similar definitions appear in textbooks even today. But the words "collection", or "group", or "aggregate", etc. do not seem to be any more comprehensible than the word "number".

The sequence of whole numbers

$$1, 2, 3, 4, 5, \ldots$$

goes on without end and is called the *set of natural numbers*.

17. The Limits of Counting

In primitive society, man could hardly count at all. He was able to distinguish groups of two and three objects, anything beyond that being thought of as "many". This was obviously not counting, it was only a beginning.

Gradually larger groups were distinguished, giving rise to the notions of "four", "five", "six", "seven". For a long time, the word "seven" was used in some languages to denote an indefinitely large quantity.

As man's activities became more intricate, the counting process developed and gave rise to a variety of reckoning devices: the making of notches on sticks and trees, knots in ropes, groups of stones, etc.*

The human hand with its five fingers was an invaluable natural tool for counting. It could not preserve the information it conveyed but it was ready at hand, so to speak, and very mobile. The language of primitive man was poor; gesticulations often made up for lack of words, and numbers (for which there were no names) were demonstrated in finger counting (this is even done nowadays if two persons speaking different languages do not understand each other).

It is quite natural that the newly originating names for "large" numbers were often constructed on the basis of the number 10 corresponding to the 10 fingers. Certain peoples developed a number system based on 5, the fingers of one hand, or on 20 which is the total number of fingers and toes (see Sec. 18).

During the early stages of man's development, the range of numbers expanded very slowly. Counting proceeded through the first tens and only much later reached one hundred. In many languages, the number 40 represented the limit and designated an indefinitely large quantity.

When the counting process reached ten tens and a name was given to the number 100, it was also used to denote an indefinitely large number (in some languages, Tartar for instance, one and the same word is used to denote 40 and 100). The very same process occurred again with the numbers thousand, ten thousand, and million.

18. The Decimal System of Numeration

In many modern languages, the names of all numbers up to a million are made up of 37 words denoting the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18,

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* Counting by means of stones (pebbles) served as the starting point for more sophisticated counting devices such as, for example, the Russian abacus, the Chinese abacus (or suan phan), the ancient Egyptian abacus (a board divided into strips where counters were placed). Other peoples had similar devices. In Latin, the idea of counting is expressed by the word "calculatio" (whence the English word "calculation") coming from "calculus", which means "pebble".

19, 20, 30, 40, 50, 60, 70, 80, 90, 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000 (for example, 918,742: nine hundred eighteen thousand seven hundred forty-two). In turn, the names of these 37 numbers are, as a rule, built up from the names of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10, 100, 1000. The underlying element of all these word formations is the number 10; that is why the modern system of numbers is called the *decimal system of numeration*. The exceptional role of 10 is due to the fact that we have 10 fingers on our two hands (see Sec. 17).

That is the general rule, but a wide variety of exceptions are evident in various languages. These are due to historical peculiarities in the development of the counting process.

In the Turkic languages (Azerbaijan, Uzbek, Turkmen, Kazakh, Tartar, Turkish, etc.) the exceptions are the names of the numbers 20, 30, 40, 50, whereas 60, 70, 80, 90 are formed on the basis of the names for 6, 7, 8 and 9. In Mongolian, on the contrary, the names of the numbers 20, 30, 40, 50 follow the general rule, while 60, 70, 80 and 90 are exceptions. In Russian there is one exception, the name for "forty". In French the names for the numbers 20 and 80 retain the nondecimal names, 80 is quatrevingt (four twenties). This is a remnant of the ancient vigesimal system of numbers (based on 20—the total number of fingers and toes). In Latin also the name for 20 is nondecimal (viginti), but that for 80 (octoginta) is decimal and comes from 8 (octo). On the other hand, the names for the numbers 18 and 19 are built up from 20 by subtraction: 20—2 and 20—1 (duodeviginti, undeviginti, that is to say, two from twenty and one from twenty). The names of the numbers 200, 300, 400, 500, 600, 700, 800, and 900 in all modern languages are constructed on the decimal (scale-10) basis.

19. Development of the Number Concept

In the counting process, unity is the smallest number. There is no need to subdivide it, nor is it even possible at times (adding half a stone to two stones yields three stones, not $2^1/_2$, and of course it is impossible to select a committee made up of $2^1/_2$ persons). However, unity often has to be broken up into parts when measuring lengths by means of steps ($2^1/_2$ steps long, and the like). For this reason, the notion of a *fractional number* (see Secs. 30 and 45) was known in remote antiquity. Subsequent development saw the expansion of the concept of number to *irrational numbers* (Sec. 91), *negative numbers* (Sec. 67), and *complex numbers* (Secs. 92 and 98).

Zero (null) was a long time in entering the family of numbers. At first, zero (null) had the meaning of absence of any number (the Latin "nullum" literally means "nothing"). If say 3 is taken away from 3 we have nothing. For that "nothing" to be considered a number, we had to wait for negative numbers to appear (see Sec. 67).

20. Numerals

A numeral is a written sign depicting a number. In the most ancient times, numbers were denoted by straight-line strokes ("rods"): one rod depicted unity, two rods, a two, and so forth. This notation originated from the use of notches. It still exists in the Roman numerals (see Sec. 21, Item 5) which denote the numbers 1, 2, 3.

This notation is inconvenient for writing large numbers and so special symbols were used to depict the number 10 (in accordance with the decimal system of numeration, see Sec. 18) and, in some languages, the number 5 as well (in accordance with quinary numeration which is based on the number of fingers of the human hand). Later, symbols were invented for still larger numbers. These symbols exhibited a variety of forms in the different languages and underwent considerable modifications in the course of time. There was also considerable variety in the *systems of numeration*, that is, modes of combining digits to form large numbers. However, in most number systems the 10-scale was pre-eminent and formed the basis of the decimal system of numeration (see Sec. 18).

21. Systems of Numeration

1. Ancient Greek numeration. The so-called *Attic system* of numeration was used in ancient Greece. The numbers 1, 2, 3, 4 were denoted by vertical strokes **I, II, III, IIII.** The number 5 had the symbol Γ (the first letter "pi", in its ancient form, of the word "pente", five); the numbers 6, 7, 8, 9 were written as 𐅃I, 𐅃II, 𐅃III, 𐅃IIII. The number 10 was depicted as Δ (the first letter of "deca", ten). The numbers 100, 1000, and 10,000 were denoted by H, X, M the initial letters of the corresponding words. The numbers 50, 500, and 5000 were given as combinations of the signs for 5 and 10, 5 and 100, 5 and 1000, namely: 𐅄, 𐅅, 𐅆. The remaining numbers, within the first ten thousand, were written as follows:

HH𐅄𐅃I=256, XX𐅄I=2051,

HHH𐅄ΔΔΔII=382, 𐅆XX𐅅HHH=7800

and so forth.

In the third century B. C., the Attic numeration gave way to the so-called *Ionian system*. Here, the numbers from 1 to 9 were denoted by the first nine letters of the alphabet (the letters ς, vau, C, koppa, and ϡ sampi, are archaic; the Greek alphabet is given in Sec. 14):

$$\alpha=1,\quad \beta=2,\quad \gamma=3,\quad \delta=4,\quad \varepsilon=5,\quad ς=6,\quad \zeta=7,\quad \eta=8,\quad \vartheta=9$$

the numbers from 10 to 90, by the next nine letters:

$$\iota=10,\quad \varkappa=20,\quad \lambda=30,\quad \mu=40,\quad \nu=50,\quad \xi=60,\quad o=70,\quad \pi=80,\quad C=90$$

the numbers from 100 to 900, by the last nine letters:

$$\rho=100,\quad \sigma=200,\quad \tau=300,\quad \upsilon=400,\quad \varphi=500;\quad \chi=600,\quad \psi=700,\quad \omega=800,\quad ϡ=900$$

Thousands and tens of thousands were denoted by the same numerals preceded by a stroke or accent:

$${}'\alpha=1000,\quad {}'\beta=2000,\quad \text{etc.}$$

A bar was placed over numerals in order to distinguish them from letters. For example,

$$\overline{\iota\eta}=18;\quad \overline{\mu\zeta}=47;\quad \upsilon\overline{\zeta}=407;\quad \overline{\chi\varkappa\alpha}=621,\quad \chi\overline{\varkappa}=620,\ \text{etc.}$$

In ancient times, the same alphabetic numeration was used by the Jews, Arabs, and many other peoples of the Near East. It is not known what people used them first.

2. **Slavic numeration.** The Slavic peoples of the south and east of Europe used the alphabetic system of notation for writing numbers. In some cases, the numerical values of the letters were established in the order of the Slavic alphabet, in others (including Russia) not all letters were used as numerals but only those found in the Greek alphabet. The letter used as a numeral was surmounted by a special symbol (see accompanying table), and the numerical values of the letters increased in the order of the letters in the Greek alphabet (the sequence of letters in the Slavic alphabet was somewhat different).

This Slavic numeration persisted in Russia till the end of the 17th century. Under Peter the First, the dominant system of numeration was the Arabic (see Item 6 below) which is still in use today. The Slavic numeration persists, however, in clerical works.

The Slavic Numerals

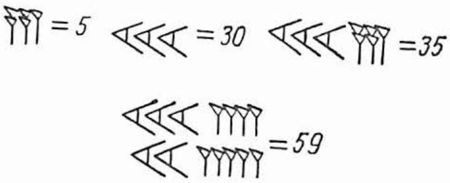
А҃	В҃	Г҃	Д҃	Є҃	Ѕ҃	З҃	И҃	Ѳ҃
1	2	3	4	5	6	7	8	9
І҃	К҃	Л҃	М҃	Н҃	Ѯ҃	О҃	П҃	Ч҃
10	20	30	40	50	60	70	80	90
Р҃	С҃	Т҃	У҃	Ф҃	Х҃	Ѱ҃	Ѡ҃	Ц҃
100	200	300	400	500	600	700	800	900

3. **The Ancient Armenian and Georgian systems of numeration.** Both Armenians and Georgians used the alphabetic principle of numeration. But the ancient alphabets of these peoples had far more letters than did the Greeks. This enab-

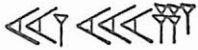
led them to use special symbols for the numbers 1000, 2000, 3000, 4000, 5000, 6000, 7000, 8000, and 9000. The numerical values of the letters followed the order of the letters in the alphabets of these peoples.

The alphabetic numeration persisted till the 18th century although the Arabic numeration was used occasionally much earlier (in the Georgian literature such instances go back to the 10 or 11th century; sources of Armenian mathematical literature reveal such usage no earlier than the 15th century). In Armenia the alphabetic numeration is still used in designations of chapters, stanzas, and the like. In Georgia the alphabetic numeration has gone out of use altogether.

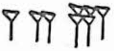
4. Babylonian Positional System of Numeration. Approximately 40 centuries prior to the Christian era, the ancient Babylonians developed a positional system of notation for their numeration. This is a mode of representing numbers in which one and the same digit is capable of denoting different numbers depending on its position. Our present-day numeration is also positional: in the number 52, the digit 5 denotes 50, that is, 5·10, while in the number 576, the same digit stands for five hundred, or 5·10·10. In Babylonian notation the number 60 was used as we use 10 in our number system, whence the name *sexagesimal* by which the Babylonian system is designated. Numbers less than 60 were denoted by two symbols: 𒁹 for unity and 𒌋 for ten. They were wedge-shaped (cuneiform) since the Babylonians wrote on clay tablets with a stylus having the form of a triangular prism. These signs were repeated as many times as needed, for example,



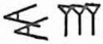
Numbers exceeding 60 were written in the following manner: 𒐊 𒐖 denoted $5\cdot 60+2=302$, rather like our notation of 52 denotes $5\cdot 10+2$. The notation



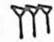
denoted the number $21\cdot 60\cdot 35=1295$. The following notation

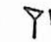
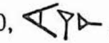
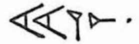


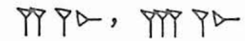
stood for $1\cdot 60\cdot 60+2\cdot 60+5=3725$; much like the modern notation 125 denotes $1\cdot 100+2\cdot 10+5$. The sign was used as a placeholder, playing the part of zero. Thus, the notation



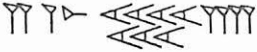
meant $2\cdot 60\cdot 60+0\cdot 60+3=7203$. But the absence of any lower order digits was not indicated; for example, the number $180=3\cdot 60$ was denoted as , which is the same as the number 3. The same notation might mean $10{,}800=3\cdot 60\cdot 60$, etc. Only the context could distinguish the numbers 3, 180, 10,800, etc.

The notation  could also signify $\frac{3}{60}$, $\frac{3}{60\cdot 60}=\frac{3}{3600}$, $\frac{3}{60\cdot 60\cdot 60}=\frac{3}{216{,}000}$ just as we use the numeral 3 to denote $\frac{3}{10}$, $\frac{3}{10\cdot 10}=\frac{3}{100}$, $\frac{3}{10\cdot 10\cdot 10}=\frac{3}{1000}$, etc. in our system of decimal fractions. However, we readily differentiate between these fractions by annexing zeros in front of the 3, and we write $\frac{3}{10}=0.3$, $\frac{3}{100}=0.03$, $\frac{3}{1000}=0.003$, etc. In the Babylonian notation these zero placeholders were not indicated.

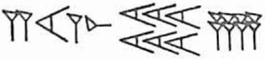
Besides the sexagesimal system of numeration, the Babylonians used the decimal system, but it was not a positional system. Apart from symbols for 1 and 10, there were symbols for 100, , 1000, , and 10,000, . The numbers 200, 300 and so on were written as



and so forth. The same method was used to write the numbers 2000, 3000 etc. 20,000, 30,000 etc. The number 274 looked like this:



the number 2068, like this:



etc.

The sexagesimal system originated at a later period than the decimal system because the numbers up to 60 were written on the basis of the decimal principle. It is still not known when and how the Babylonians developed the sexagesimal system. There are numerous hypotheses as to how this occurred but there is no firm proof for any of them.

The sexagesimal notation of whole numbers did not spread beyond the Assyrian-Babylonian empire, but sexagesimal fractions spread far and wide to the countries of the Near East, Central Asia, Northern Africa, and Western Europe. They found wide use, especially in astronomy, right up to the invention of decimal fractions (which was at the beginning of the 17th century). Traces of sexagesimal fractions are still found in the divisions of the degree of angle and arc (and also the hour) into 60 minutes, and of the minute into 60 seconds.

5. Roman numeration. The ancient Romans used a number system that is still in use and is called the Roman system of numeration. We use it for designating congresses and conferences, for numbering the introductory pages of books, chapter headings, etc.

In their latest form, the Roman numerals looked like this:

I = 1, V = 5, X = 10, L = 50, C = 100, D = 500, M = 1000

The earlier forms were somewhat different. Thus, the number 1000 was denoted by the symbol (|), 500 by the symbol |).

There is no reliable information on the origin of the Roman numerals. The numeral V might have originally depicted the human hand, the numeral X could have been built up out of two fives. In the same way, the symbol for 1000 could have developed out of doubling the sign for 500 (or vice versa).

The Roman system of numeration exhibits evident traces of the quinary system of numbers. But Latin (the language of the Romans) does not have a trace of the quinary number system. This must mean that these numerals were borrowed by the Romans from another people (most likely from the Etruscans).

All whole numbers (up to 5000) are written by means of iteration of the numerals listed above. If a large numeral precedes a smaller one, they are added, if the smaller one comes first (in which case the symbol is not repeated), then it is subtracted from the larger numeral (in Latin, the subtractive principle—see Sec. 18—is preserved in the names of two cardinal numbers: 18 and 19). For example, VI = 6, or 5 + 1, IV = 4, or 5 − 1, XL = 40, or 50 − 10, LX = 60, or 50 + 10. No digit is repeated more than three times: LXX = 70, LXXX = 80; the number 90 is written as XC (and not LXXXX).

The first 12 numbers are written in Roman numerals as follows:

I, II, III, IV, V, VI, VII, VIII, IX, X, XI, XII

Examples: XXVIII = 28, XXXIX = 39, CCCXCVII = 397, MDCCCXVIII = 1818

Performing arithmetical operations with multidigit numbers is an arduous task when done in Roman numerals. Nevertheless, Roman numerals were still the dominant number system in Italy up to the 13th century, and in other countries of Western Europe they persisted till the 16th century.

6. The positional numeration of India. The various regions of India had different number systems, one of which spread to other parts of the world and is today the generally accepted system of numeration. In this system, the numerals had the forms of the initial letters of the appropriate cardinal numbers in the ancient Indian language of Sanskrit (the Devanagari alphabet).

Originally, these symbols denoted the numbers 1, 2, 3, ..., 9, 10, 20, 30, ..., 90, 100, 1000, which in turn were used to write the other numbers. Later, a special sign (a heavy dot, circle) was introduced to indicate an empty position in a number. The signs for numbers exceeding 9 ceased to be used at a later period, and the Devanagari system of numeration became the decimal positional system of numeration. It is not known how and when this conversion took place, but by the middle of the 8th century, the positional system of numeration was in wide use in India. It was about this time that it began to spread to other countries (Indochina, China, Tibet, into the territory of the present-day Central-Asian republics of the Soviet Union, Iran, and elsewhere). A decisive role in the spread of the Hindu numeration in the Arabic countries was played by a manual written at the beginning of the 9th century by Mohammed ibn-Musa al-Khowarizmi (from Khoresm—the present-day Khoresm Oblast of the Uzbek Republic of the USSR). * It was translated into Latin in Western Europe in the 12th century. In the 13th century, the Hindu system of numeration became dominant in Italy, and by the 16th century it spread to the other countries of Western Europe. The Europeans borrowed the Hindu number system from the Arabs and called it the Arabic system of numeration. Historically, this is not correct, but the name persists.

The Arabs also gave us the word "cipher" ("sifr" in Arabic) which literally means "empty position" (this is a translation of the Sanskrit "sunia" which has the same meaning).

The word was originally used to denote the empty position (as a placeholder) in a number and that meaning was still current in the 18th century, although in the 15th century the Latin term for "zero", "null" (nullum—nothing), had appeared.

The shapes of the Hindu numerals underwent a variety of modifications over the centuries; the form that we have today was established in the 16th century.

22. Names of Large Numbers

To facilitate reading and remembering large numbers, the digits are ordinarily grouped into periods of three each, which are separated by a comma or a space. Say, the number 35461298 is written: 35 461 298 (or 35,461,298). Here, the three digits 298 form the first period, 461, the second, and 35, the third.

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* This remarkable scholar was also the founder of algebra (see Sec. 66). Mohammed wrote his works in Arabic, which in the East was the common language of learning, just as Latin was at one time in Western Europe. This explains the name al-Khowarizmi (which means a dweller of Khoresm) by which Mohammed is known in history.

Each digit has place value. From right to left we have: 8 is the units digit, 9 the tens digit, 2 the hundreds digit, 1 the thousands digit, 6 the ten thousands digit, etc.

The first period yields units, tens, hundreds, the second period, thousands, and the third period, millions.

The American system of numeration for denominations above one million is the same as the French system and the Russian system, and the British system corresponds to the German. In the American system each of the denominations above 1000 millions (the American billion) is 1000 times the one preceding (one trillion = 1000 billions; one quadrillion = 1000 trillions, one quintillion = 1000 quadrillions, one sextillion = 1000 quintillions, and so on for septillions, octillions, nonillions, etc.). In the British system the first denomination above 1000 millions (the British milliard) is 1000 times the preceding one, but each of the denominations above 1000 milliards (the British billion) is 1,000,000 times the preceding one (one trillion = 1,000,000 billions; one quadrillion = 1,000,000 trillions, etc.).

23. Arithmetic Operations

1. **Addition.** The concept of adding stems from such fundamentally elementary facts that it does not require a definition and cannot be defined in formal fashion. We can use synonymous expressions, if one so desires, and say it is the process of combining, and the like.

Notation: $8+3=11$; 8 and 3 are the *addends*, 11 is the *sum*.

2. **Subtraction.** When one number is subtracted from another the result is called the *difference* or *remainder*. The number subtracted is termed the *subtrahend*, and the number from which the subtrahend is subtracted is called the *minuend*.

Notation: $15-7=8$; 15 is the minuend, 7 is the subtrahend, and 8 is the remainder. Subtraction may be checked by addition: $8+7=15$.

3. **Multiplication.** Multiplication is the process of taking one number (called the *multiplicand*) a given number of times (this is the *multiplier*, which tells us how many times the multiplicand is to be taken). The result is called the *product*. The numbers multiplied together are called the factors of the product. (For multiplication by a fraction see Sec. 34).

Notation: $12\times5=60$ or $12\cdot5=60$; 12 is the multiplicand, 5 the multiplier, and 60 the product (here, 12 and 5 are the factors of the product). $12\times5=12+12+12+12+12$.

Nothing changes if we call 5 the multiplicand and 12 the multiplier. To take another example, $2\times5=2+2+2+2+2=10$ and $5\times2=5+5=10$. It is therefore more common to use the generic term "factor" for the multiplier and the multiplicand.

4. Division. Division is the process of finding one of two factors from the product and the other factor. It is the process of determining how many times one number is contained in another. The number divided by another is called the *dividend*. The number divided into the dividend is called the *divisor*, and the answer obtained by division is called the *quotient*.

Notation: $48:6=8$ (or $48\div6=8$). Here, 48 is the dividend, 6 the divisor, and 8 the quotient. Division may be checked by multiplication: the product of the divisor (6) and the quotient (8) yields the dividend. Division may also be written as $\frac{48}{6}=8$ (see Sec. 36).

The quotient obtained by the division of one whole number by another one may not be a whole number, in which case the quotient may be indicated as fraction (Sec. 30). If the quotient is a whole number, we say that the first number is exactly divisible (or, simply, divisible) by the second number. For example, 35 is exactly divisible by 5 since the quotient is a whole number, 7.

In this case, the second number (5) is called the divisor of the first one, and the first number (35) is termed the multiple of the second one.

Example 1. 5 is a divisor of the numbers 25, 60, 80 and is not a divisor of 4, 13, 42, or 61.

Example 2. 60 is a multiple of 15, 20, 30 and is not a multiple of 17, 40, or 90.

In many cases it is possible to determine whether one number is divisible by another without actually performing the division (see Sec. 25).

In most cases there is a remainder after the division. In the process of division with a remainder we seek the largest whole number which, when multiplied by the divisor, yields a number that does not exceed the dividend. The desired number is a *partial quotient*. The difference between the dividend and the product of divisor by the partial quotient is termed the *remainder*, which is always less than the divisor.

Example. 19 is not divisible exactly by 5. The numbers 1, 2, 3 when multiplied by 5 yield 5, 10, 15, none of which exceeds 19, whereas 4 by 5 yields 20, which exceeds 19. Hence, the partial quotient is 3. The difference between 19 and the product $3\cdot 5=15$ is $19-15=4$. The remainder is 4.

For division by zero, see Sec. 37.

5. **Involution** (raising a number to a power). To raise a number to an integral power (second, third, fourth, etc.) we multiply the quantity by itself two, three, four, etc. times (for raising a number to a negative, zero or fractional power, see Sec. 125). The number repeated as a factor is called the *base*; the number which indicates how many times the base is to be used as a factor is called the *exponent of the power*. The result is called the *power*.

Notation: $3^4=81$. Here, 3 is the base, 4 is the exponent, and 81 is the power; $3^4=3\cdot 3\cdot 3\cdot 3$.

The second power is also called the *square*, the third power is also called the *cube* (of a number). The first power of a number is the number itself.

6. **Evolution** (finding the root of a number). Finding (or extracting) the root of a number is a process by which we find the base of a power from the power and the exponent. The power here is termed the *radicand*, the exponent is here the *index of the root*, and the desired base of the power is termed the *root*.

Notation: $\sqrt[4]{81}=3$. Here, 81 is the radicand, 4 the index, and 3 the root. Raising 3 to the fourth power yields 81; $3^4=81$ (evolution can be checked by involution).

The second root is usually called the *square root*, the third root the *cube root*. When taking square roots the index 2 is usually omitted: $\sqrt{16}=4$ means $\sqrt[2]{16}=4$.

Addition and subtraction, multiplication and division, involution and evolution are pairs of *inverse operations*.

It is assumed that the reader is acquainted with the rules of the first four operations with whole numbers. Raising to a power is performed by means of repeated multiplication. For extraction of roots (evolution), see Secs. 58, 58a.

24. Order of Operations. Brackets

If several operations are performed one after another, the result, generally speaking, depends on the sequence (order) of the operations. For example, $4-2+1=3$ if the operations are performed as indicated; but if we first add 2 and 1 and then subtract the sum from 4, we get 1. *Brackets* are used to indicate the sequence in which the operations are to be performed (in cases when the result depends on the order of the operations). Operations indicated in brackets are to be performed first. In our case, $(4-2)+1=3$, or $4-(2+1)=1$.

Example 1. $(2+4)\times5=6\times5=30$; $2+(4\times5)=2+20=22$.

In order to avoid too many brackets, we agree not to indicate brackets: (1) when the operations of addition and subtraction are to be performed in the sequence in which they are indicated; for instance, in place of $(4-2)+1=3$ we write $4-2+1=3$; (2) when multiplication or division would be indicated in brackets; for instance, in place of $2+(4\times5)=22$ we write $2+4\times5=22$.

In computing expressions which either have no brackets or contain only one set of brackets, perform the operations in the following order: (1) first the operations in the brackets; multiplication and division in the sequence in which they are given but before addition and subtraction; (2) then the remaining operations, again multiplication and division being accomplished in the order in which they are indicated but prior to addition and subtraction.

Example 2. $2\cdot5-3\cdot3$. First multiply, $2\cdot5=10$, $3\cdot3=9$, then subtract $10-9=1$.

Example 3. $9+16:4-2\cdot(16-2\cdot7+4)+6\cdot(2+5)$. First perform the operations indicated in the brackets:

$$16-2\cdot7+4=16-14+4=6;\ 2+5=7$$

Now handle the remaining operations:

$$9+16:4-2\cdot6+6\cdot7=9+4-12+42=43$$

It often happens that bracketed expressions themselves have to be enclosed in brackets, and the latter once more in brackets. In such cases, the *round brackets (parentheses)* are used first, then *square brackets*, [ ], and finally *curly brackets* (braces), { }. The sequence of operations then is as follows: perform all operations in the round brackets in the indicated order; then all computations in the square brackets by the same rules, and all the computations in the curly brackets, etc.; finally, the remaining operations are performed.

Example 4. $5+2\times[14-3\cdot(8-6)]+32:(10-2\cdot3)$. We carry out the operations in the round brackets: this yields

$$8-6=2;\quad 10-2\cdot3=10-6=4$$

the operations in the square brackets yield $14-3\cdot2=8$, and then the remaining operations

$$5+2\cdot8+32:4=5+16+8=29$$

which give us our answer.

Example 5. $\{100-[35-(30-20)]\}\cdot2$. The sequence of operations is: $30-20=10$; $35-10=25$; $100-25=75$; $75\cdot2=150$.

25. Criteria for Divisibility

Divisibility by 2. A number divisible by 2 is called an *even number*, otherwise it is *odd*. A number is divisible by two if its last digit is even or zero, otherwise it is not.

Example. The number 52,738 is divisible by 2 since the last digit is 8 (even); 7691 is not divisible by 2 because the last digit, 1, is odd; 1250 is divisible by 2 because the last digit is zero.

Divisibility by 4. A number is divisible by 4 if the last two digits are zeros or form a number divisible by 4, otherwise it is not so divisible.

Examples. 31,700 is divisible by 4 since the last two digits are zeros; 215,634 is not divisible by 4 because the last two digits form the number 34, which is not divisible by 4; 16,608 is divisible by 4 since the last two digits 08 yield the number 8 which is divisible by 4.

Divisibility by 8. A number is divisible by 8 if the last three digits are zeros or form a number divisible by 8. Otherwise it is not divisible by 8.

Examples. 125,000 is divisible by 8 (the last three digits are zeros); 170,004 is not divisible by 8 (the last three digits form the number 4, which cannot be divided by 8); 111,120 is divisible by 8 since the last three digits form 120, which is divisible by 8.

Similar criteria could be indicated for division by 16, 32, 64, etc., but they are of no practical value.

Divisibility by 3 and by 9. Only such numbers are divisible by 3 the sum of whose digits is divisible by 3; the same for 9—the sum of the digits must be divisible by 9.

Examples. The number 17,835 is divisible by 3 but is not divisible by 9 since the sum of the digits $1+7+8+3+5=24$ is divisible by 3 but is not divisible by 9. The number 106,499 is not divisible either by 3 or by 9 since the sum of the digits (29) is not divisible by 3 or 9. The number 52,632 is divisible by 9 because the sum of the digits (18) is divisible by 9.

Divisibility by 6. A number is divisible by 6 if it is simultaneously divisible by 2 and by 3, otherwise it is not.

Example. 126 is divisible by 6 since it can be divided by 2 and by 3.

Divisibility by 5. Numbers ending in 0 or 5 are divisible by 5, otherwise they are not.

Example. 240 is divisible by 5 (the last digit is zero); 554 is not divisible by 5 (the last digit is 4).

Divisibility by 25. A number is divisible by 25 if the last two digits are zeros or form a number that is divisible by 25 (such are 00, 25, 50, or 75), otherwise it is not divisible by 25.

Example. 7150 is divisible by 25 (it ends in 50); 4855 is not divisible by 25.

Divisibility by 10, by 100 and by 1000. Only numbers ending in 0 are divisible by 10, only those ending in two zeros are divisible by 100, and only those ending in three zeros are divisible by 1000.

Examples. 8200 is divisible by 10 and by 100; 542,000 is divisible by 10, 100 and 1000.

Divisibility by 11. The number 11 divides only those numbers whose sum of digits occupying odd positions is either equal to the sum of the digits occupying even positions or differs from it by a number which is divisible by 11.

Examples. The number 103,785 is divisible by 11 since the sum of the digits in odd positions $(1+3+8=12)$ is equal to the sum of the digits in even positions $(0+7+5=12)$. The number 9,163,627 is divisible by 11 since the sum of digits in odd positions is $9+6+6+7=28$, while the sum of digits in even positions is $1+3+2=6$; the difference between 28 and 6 is 22, which is divisible by 11. The number 461,025 is not divisible by 11 since the numbers $4+1+2=7$ and $6+0+5=11$ are not equal and their difference $11-7=4$ is not divisible by 11.

There exist divisibility criteria for other numbers besides those given above but they are more complicated.

26. Prime and Composite Numbers

All whole numbers (integers) other than 1 have at least two divisors: unity (one) and the number itself. Numbers that do not have any other divisors are called *prime numbers*. For example, 7, 41, 53 are prime numbers. Numbers which have other divisors besides 1 and the number itself are called *composite numbers*. An example is 21 with its divisors 1, 3, 7, 21 and 81 with its divisors 1, 3, 9, 27, 81. The number one (unity) could be classed as a prime, but it is better to put it in a separate class outside the prime and composite numbers (this convention stems from the fact that many of the rules which hold true for all other primes are invalid when applied to unity).

There exist infinitely many prime numbers.

The primes not exceeding 200 are (see pages **50-51** for the Table of Primes < 6000):

$$\left.\begin{array}{l} 2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 37,\ 41,\ 43, \\ 47,\ 53,\ 59,\ 61,\ 67,\ 71,\ 73,\ 79,\ 83,\ 89,\ 97,\ 101, \\ 103,\ 107,\ 109,\ 113,\ 127,\ 131,\ 137,\ 139,\ 149,\ 151, \\ 157,\ 163,\ 167,\ 173,\ 179,\ 181,\ 191,\ 193,\ 197,\ 199 \end{array}\right\} \quad (A)$$

27. Factorization into Prime Factors

Every composite number can be represented uniquely as a product of prime factors. For example, $36 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2$; $45 = 3 \cdot 3 \cdot 5 = 3^2 \cdot 5$ (or $3^2 \cdot 5^1$), $150 = 2 \cdot 3 \cdot 5 \cdot 5 = 2 \cdot 3 \cdot 5^2$ (or $2^1 \cdot 3^1 \cdot 5^2$). For small numbers, factorization can be accomplished by mere guesswork. For large numbers the following technique can be used.

Example 1. Suppose we have a number 1421. Taking the primes, one after the other, out of Table of Primes (A), we stop at the prime which is a divisor of the given number. Using the divisibility criteria we see that the numbers 2, 3, 5 cannot be divisors of 1421; we attempt to divide by 7 and we see that 1421 is divisible by 7 yielding a quotient of 203. To the left of the line we write 1421; on the right the divisor; under the number we write the quotient 203. Then we test 203 in the same manner. We ignore the numbers 2, 3, 5 that proved ineffective in the first trial and begin with 7. It turns out that 7 is a divisor of 203. Write it down to the right of 203. Under 203 write the quotient 29, which

Work:

$$\begin{array}{r|l} 1421 & 7 \\ 203 & 7 \\ 29 & 29 \end{array}$$

is prime. This completes the factorization. We have:

$$1421 = 7 \cdot 7 \cdot 29 = 7^2 \cdot 29$$

The general technique may be simplified in a number of cases.

Example 2. Let us factor the number 1,237,600 into prime factors. Noting that $1{,}237{,}600 = 12{,}376 \times 100$, we factor these two factors separately. The latter is readily decomposable into $100 = 10 \cdot 10 = 2 \cdot 5 \cdot 2 \cdot 5 = 2^2 \cdot 5^2$. Then we factor the first one as follows.

Work:		
	12,376	2
	6 188	2
	3 094	2
	1 547	7
	221	13
	17	17

From Table (A) we take the first prime, 2; we immediately see that it is a divisor of 12,376. Finding the quotient 6188, we again take 2 from Table (A). The second quotient 3094 is also even, so we divide it by 2. The result 1547 is no longer divisible by 2. The divisibility criteria show that it is not divisible by 3 or by 5 either. Let us try 7. We get the quotient 221. Try 7 again. It is not divisible. Then try the following primes. 221 is not divisible by 11 but it is divisible by 13 with the quotient 17, which is prime.

The final result is: $1{,}237{,}600 = 2^3 \cdot 7 \cdot 13 \cdot 17 \cdot 2^2 \cdot 5^2 = 2^5 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$.

28. Greatest Common Divisor

A *common divisor* of several numbers is a divisor (see Sec. 23, Item 4) of each of them. For example, the numbers 12, 18, 30 have 3 as a common divisor; 2 is also a common divisor. In every set of common divisors there is always a *greatest common divisor* (6 in our example), or G.C.D.

Examples. 4 is the G.C.D. of 16, 20, 28; 5 is the G. C. D. of 5, 30, 60, and 90.

For small numbers, the G.C.D. can be found by inspection. If we have to do with large numbers, decompose each into prime factors (see Sec. 27) and write out those which are factors of all the given numbers. Take each factor with the smallest exponent with which it is contained in the given numbers. Then multiply.

Example 1. Find the G.C.D. of 252, 441, and 1080. Factor into prime factors.

$$252 = 2^2 \cdot 3^2 \cdot 7, \quad 441 = 3^2 \cdot 7^2, \quad 1080 = 2^3 \cdot 3^3 \cdot 5$$

We have only one common factor, the prime factor 3. The smallest exponent of this factor is 2. The G.C.D. is $3^2=9$.

Example 2. Find the G.C.D. of the numbers 234, 1080, and 8100. $234=2\cdot3^2\cdot13$, $1080=2^3\cdot3^3\cdot5$, $8100=2^2\cdot3^4\cdot5^2$

$$\text{G. C. D.}=2\cdot3^2=18$$

It may happen that there are no prime factors common to all the numbers. Then the greatest common divisor is 1. For instance, for $15=3\cdot5$, $10=2\cdot5$, $6=2\cdot3$ the G.C.D. is 1. If the G.C.D. of two numbers is 1, then these numbers are called relatively prime. To illustrate, 15 and 22 are relatively prime numbers.

29. Least Common Multiple

A *common multiple* of several numbers is a multiple of each of them (Sec. 23, Item 4). The numbers 15, 6, and 10 have 180 as a common multiple. The number 90 is also a common multiple of these numbers. The set of common multiples has a *least common multiple* (30 in our case), or L.C.M. When dealing with small numbers, the L.C.M. can be seen at once. If the number is large, do as follows: factor the given numbers into prime factors; write out all prime factors of at least one of the given numbers; then take each factor and raise it to the highest power that it is contained in the given numbers. Then multiply.

Example 1. Find the L.C.M. of 252, 441, 1080.

Factor into prime factors: $252=2^2\cdot3^2\cdot7$, $441=3^2\cdot7^2$, $1080=2^3\cdot3^3\cdot5$. Multiply out $2^3\cdot3^3\cdot7^2\cdot5$ and we find the L.C.M. to be 52,920.

Example 2. Find the L.C.M. of 234, 1080, 8100 (see Sec. 28, Example 2). The L.C.M. $=2^3\cdot3^4\cdot5^2\cdot13=210{,}600$.

30. Common Fractions

A *common fraction* (or, simply, a fraction) is a part of unity or several equal parts of unity. The number which indicates how many parts a unit is divided into is called the *denominator* of the fraction; the number indicating how many parts are taken is the *numerator* of the fraction.

Notation: $\frac{3}{5}$ or 3/5 (three fifths): here, 3 is the numerator and 5 is the denominator.

If the numerator is less than the denominator, the fraction is less than unity and is called *a proper fraction:* $\frac{3}{5}$ is a proper fraction. If the numerator is equal to the denominator, the fraction is equal to unity. If the numerator is greater than the denominator, the fraction exceeds unity. In these latter two cases it is called an *improper fraction*. For example, $\frac{5}{5}$, $\frac{17}{5}$ are improper fractions. In order to take out the largest whole number contained in an improper fraction, divide the numerator by the denominator. If the division is exact, then the improper fraction is equal to the quotient. Say, $\frac{45}{5}=45:5=9$. If the division is not exact, the (partial) quotient yields the desired integer, while the remainder becomes the numerator of the fractional part, and the denominator of the fractional part remains unchanged.

Example. Given the fraction $\frac{48}{5}$. Divide 48 by 5 to get the quotient 9 and a remainder of 3; $\frac{48}{5}=9\frac{3}{5}$.

A number consisting of an integral part (whole number) and a fractional part (fraction) is called a *mixed number* $\left(9\frac{3}{5}\text{ for instance}\right)$. The fractional part of a mixed number may be an improper fraction $\left(\text{like } 7\frac{13}{5}\right)$. It is then possible to take out the largest whole number (see above) and represent the mixed number so that the fraction becomes proper (or disappears altogether). For example,

$$7\frac{13}{5}=7+\frac{13}{5}=7+2\frac{3}{5}=9\frac{3}{5}$$

Mixed numbers are usually reduced to such form.

If is often necessary (in multiplying fractions, say) to operate in reverse: given a mixed number, it is required to represent it in the form of a fraction (improper fraction). To do this, (1) multiply the integer in the mixed number by the denominator of the fractional part; (2) add the numerator to the product. The resulting number will be the numerator of the desired fraction, and the denominator remains unchanged.

Example. Given the mixed number $9\frac{3}{5}$. (1) $9 \cdot 5 = 45$; (2) $45 + 3 = 48$; (3) $9\frac{3}{5} = \frac{48}{5}$

31. Reducing Fractions

The value of a fraction is not changed if the numerator and denominator are multiplied by the same number. For example

$$\frac{3}{5} = \frac{3 \cdot 6}{5 \cdot 6} = \frac{18}{30}, \quad \frac{1}{2} = \frac{1 \cdot 3}{2 \cdot 3} = \frac{3}{6}, \quad \frac{1}{2} = \frac{1 \cdot 4}{2 \cdot 4} = \frac{4}{8}$$

This is called *reducing (changing) the fraction to higher terms.*

The value of a fraction remains unchanged if the numerator and denominator are divided by one and the same number. For example

$$\frac{18}{30} = \frac{18:6}{30:6} = \frac{3}{5}, \quad \frac{4}{8} = \frac{4:4}{8:4} = \frac{1}{2}$$

This is called *reducing the fraction to lower terms*, or simply reducing the fraction. We say that $\frac{3}{5}$ is obtained from $\frac{18}{30}$ by dividing out 6.

A fraction can only be reduced if the numerator and denominator have the same divisors (that is, if they are not relatively prime). The reduction may be accomplished gradually or at once using the G.C.D.

Example. Reduce the fraction $\frac{108}{144}$ to lower terms. Using the divisibility-by-4 criterion (see Sec. 25 above), we see that 4 is a common divisor of the numerator and denominator. Dividing out 4 we have $\frac{108}{144} = \frac{108:4}{144:4} = \frac{27}{36}$. Noting that 27 and 36 have 9 as a common divisor, we divide 9 out of $\frac{27}{36}$; $\frac{27}{36} = \frac{3}{4}$. Further reduction is impossible since 3 and 4 are relatively prime.

We get the same result if we find the greatest common divisor of the numbers 108 and 144. It is 36. Dividing out 36 we obtain

$$\frac{108}{144} = \frac{108:36}{144:36} = \frac{3}{4}$$

Dividing by the greatest common divisor we have the *fraction in lowest terms.*

32. Comparing Fractions.

Finding a Common Denominator

If two fractions have the same numerator, the larger fraction is that with the smaller denominator. For example, $\frac{1}{3} > \frac{1}{4}$, $\frac{5}{7} > \frac{5}{9}$. If two fractions have the same denominator, the greater fraction is that with the greater numerator: $\frac{5}{8} > \frac{3}{8}$.

In order to compare two fractions with different numerators and denominators, either one or both of the fractions have to be transformed so that the denominators are the same. To do this, change the first fraction to the terms of the denominator of the second, and the second to the denominator of the first.

Example. Compare the fractions $\frac{3}{8}$ and $\frac{7}{12}$. Change the first by a factor of 12, the second by 8. This yields $\frac{3}{8}=\frac{36}{96}$, $\frac{7}{12}=\frac{56}{96}$. Now the denominators are the same. Comparing the numerators we see that the second fraction is greater than the first.

This process of changing fractions is called *finding a common denominator*.

To reduce several fractions to a common denominator, we can change the denominator of each by a factor equal to the product of the other denominators. For instance, in order to reduce the fractions $\frac{3}{8}$, $\frac{5}{6}$, $\frac{2}{5}$ to a common denominator, change the first denominator by $5 \cdot 6 = 30$, the second by $8 \cdot 5 = 40$, the third by $8 \cdot 6 = 48$. We get $\frac{3}{8}=\frac{90}{240}$, $\frac{5}{6}=\frac{200}{240}$, $\frac{2}{5}=\frac{96}{240}$. The common denominator is the product of the denominators of all the given fractions: $8 \cdot 6 \cdot 5 = 240$.

This method of finding a common denominator is the simplest and, in many cases, the most practical. The sole inconvenience is that the common denominator may turn out to be too big, whereas smaller ones may be available. Namely, for a common denominator we can take any common multiple (the least common multiple, say) of the given denominators. It is then necessary to change (multiply) each fraction by the quotient obtained from dividing the common multiple by the denominator of the given fraction (this quotient will be called the *additional factor*).

Example. Given the fractions $\frac{3}{8}$, $\frac{5}{6}$, $\frac{2}{5}$. The least common multiple of the denominators 8, 6, 5 is 120. The additional factors are: 120:8 = 15, 120:6 = 20, 120:5 = 24. Change (multiply) the first fraction by 15, the second by 20, the third by 24:

$$\frac{3}{8}=\frac{45}{120},\quad \frac{5}{6}=\frac{100}{120},\quad \frac{2}{5}=\frac{48}{120}$$

Most arithmetics give only this technique for reducing to a common denominator. Actually, it is of practical use only when the L.C.M. is seen by inspection, otherwise a great deal of time is needed to find the L.C.M. and the additional factors. What is more, it often happens that the L.C.M. is not much less than the product of the denominators or not less at all. But then a lot of time is spent for no good reason.

33. Adding and Subtracting Fractions

If the denominators of the fractions are the same, the fractions may be added by adding their numerators; to subtract them, subtract the numerator of the subtrahend from the numerator of the minuend. This sum or difference will be the numerator of the answer, the denominator remaining unchanged. If the denominators differ, first reduce the fractions to a common denominator.

Example 1. $\frac{5}{8}+\frac{7}{8}=\frac{12}{8}=1\frac{4}{8}=1\frac{1}{2}$.

Example 2. $\frac{3}{8}+\frac{5}{6}-\frac{2}{5}=\frac{45}{120}+\frac{100}{120}-\frac{48}{120}=\frac{97}{120}$.

To add mixed numbers, separately find the sum of the integral parts and the fractional parts.

Example 3. $7\frac{3}{4}+4\frac{5}{6}=(7+4)+\left(\frac{3}{4}+\frac{5}{6}\right)=11\frac{19}{12}$
$=12\frac{7}{12}$.

When subtracting mixed numbers, the fractional part of he subtrahend may be larger than the fractional part of the minuend. In that case, borrow unity from the minuend and change to an improper fraction.

Example 4. $7\frac{1}{4}-4\frac{1}{3}=7\frac{3}{12}-4\frac{4}{12}=6\frac{15}{12}-4\frac{4}{12}=2\frac{11}{12}$.

Example 5. $11-10\frac{5}{7}=10\frac{7}{7}-10\frac{5}{7}=\frac{2}{7}$.

34. Multiplication of Fractions. Definition

For multiplication and division of a fraction by a whole number (integer), the definitions given above in Sec. 23 (Items 3 and 4) hold true. For example,

$$2\frac{3}{4}\times 3=2\frac{3}{4}+2\frac{3}{4}+2\frac{3}{4}=8\frac{1}{4}$$

Conversely, $8\frac{1}{4}:3=2\frac{3}{4}$. The practical rules of computation are given below.

The definition of Sec. 23 is not valid for multiplication by a fraction. For example, the operation $2\frac{1}{2}\cdot\frac{3}{4}$ cannot be carried out if it is understood that $2\frac{1}{2}$ is to be taken as a factor $\frac{3}{4}$ times.

To multiply a given number (integer or fraction) by a fraction means to divide the number by the denominator of the fraction and multiply the result by the numerator.

Example. $800\cdot\frac{3}{4}$, $800:4=200$ $200\cdot 3=600$ so that $800\cdot\frac{3}{4}=600$. The sequence of operations (division and multiplication) may be reversed and the result will be the same: $800\cdot 3=2400$ and $2400:4=600$.

The above definition is not a mere whim, it follows from the necessity to preserve intact the role that multiplication plays in practice and theory when we deal with whole numbers. Two examples will suffice to make this clear.

Example. A litre of kerosene weighs 800 grams. What is the weight of 4 litres? **Solution:** $800\cdot 4=3200$(g) $=3$ kg 200 g. The result is found by multiplying by 4.

What does $\frac{3}{4}$ of a litre of kerosene weigh? **Solution:** $800\cdot\frac{3}{4}=600$ (g) (see preceding example).

If we define multiplication by a fraction differently, the answer will not be correct. If we proceeded from the definition given in Sec. 23 and regarded multiplication by $\frac{3}{4}$ as impossible, then we would have to solve the problem of the weight of the kerosene by different operations: multiplication for an integral number of litres, and via a different operation for a fractional number of litres.*

In the multiplication of whole numbers, the product remains the same no matter what the order of the factors: $3\cdot4=4\cdot3=12$. This property is preserved in multiplication by a fraction as well. For example, $\frac{2}{3}\cdot3=\frac{2}{3}+\frac{2}{3}+\frac{2}{3}=2$. This result is obtained on the basis of the earlier definition (see Sec. 23). Interchange the factors: $3\cdot\frac{2}{3}$. The latter definition of multiplication no longer holds true, but the new definition yields $3\cdot\frac{2}{3}=2$.

Generally speaking, it turns out that under the new definition of multiplication, all earlier properties and rules remain valid, with the exception of one: in the earlier definition of multiplication, a number increased in value, whence the name "multiplication" ("multus" = many). Now we have to say that in multiplication by a number exceeding unity the multiplicand increases, and in multiplication by a number less than unity (a proper fraction), the multiplicand decreases. This discrepancy between the latter fact and the name of the operation is due to the fact that the term "multiplication" originated at so remote a time that the concept of multiplication referred solely to whole numbers.

35. Multiplication of Fractions. Rule

To multiply a fraction by a fraction, multiply the numerators together for the numerator of the product and multiply the denominators together for the denominator of the product. If there are mixed numbers, convert them to improper fractions before multiplying. Also, before multiplying, divide out (cancel) any common factors in the numerator and the denominator.

---

* The question naturally arises as to whether it is possible to give a definition suitable for multiplication by an integer and by a fraction. This appears to be impossible: when defining multiplication by a fraction we unavoidably have to assume as known multiplication by a whole number (see definition of this section).

Example 1. $2\frac{1}{12}\cdot 1\frac{7}{20}=\frac{25}{12}\cdot\frac{27}{20}=\frac{5}{4}\cdot\frac{9}{4}=\frac{45}{16}=2\frac{13}{16}$ (5 is divided out of 25 and 20, 3 out of 12 and 27).

The foregoing is extended to the case when the number of factors exceeds two.

Example 2. $4\frac{1}{2}\cdot\frac{4}{7}\cdot 4\frac{2}{3}=\frac{9\cdot 4\cdot 14}{2\cdot 7\cdot 3}=\frac{3\cdot 2\cdot 2}{1\cdot 1\cdot 1}=12$ (3 is divided out of 9 and 3, 2 out of 4 and 2, 7 out of 14 and 7).

If some of the factors are whole numbers, then each one is regarded as a fraction with denominator 1.

Example 3. $\frac{5}{8}\cdot 7\cdot\frac{4}{15}=\frac{5\cdot 7\cdot 4}{8\cdot 1\cdot 15}=\frac{1\cdot 7\cdot 1}{2\cdot 1\cdot 3}=\frac{7}{6}=1\frac{1}{6}$ (5 is divided out of 5 and 15, 4 out of 4 and 8).

36 Division of Fractions

The definition of division given earlier in Sec. 23 holds true for the division of fractions as well. From it follows the rule:

To divide a number by a fraction, multiply the number by the reciprocal of the fraction (the reciprocal is the fraction formed by interchanging the numerator and denominator: the reciprocal of $\frac{6}{7}$ is $\frac{7}{6}$).

Example 1. $\frac{2}{3}:\frac{4}{15}$. The reciprocal of $\frac{4}{15}$ is $\frac{15}{4}$. Hence $\frac{2}{3}:\frac{4}{15}=\frac{2\cdot 15}{3\cdot 4}=2\frac{1}{2}$.

Example 2. $1\frac{3}{5}:3\frac{1}{5}=\frac{8}{5}:\frac{16}{5}=\frac{8\cdot 5}{5\cdot 16}=\frac{1\cdot 1}{1\cdot 2}=\frac{1}{2}$.

This rule is also applicable when the dividend and divisor are whole numbers. For example, $2:5=2\cdot\frac{1}{5}=\frac{2}{5}$. Thus, the fraction bar is equivalent to the division sign.

37. Operations Involving Zero

Addition. Adding zero to any number leaves the number unchanged: $5+0=5$; $3\frac{5}{7}+0=3\frac{5}{7}$.

Subtraction. Subtracting zero from any number leaves the number unchanged: $5-0=5$; $3\frac{5}{7}-0=3\frac{5}{7}$.

Multiplication. Zero times any number is always zero: $5\cdot 0=0$; $0\cdot 3\frac{5}{7}=0$; $0\cdot 0=0$.

Division. 1. The quotient obtained by the division of zero by any nonzero number is zero: $0:7=0$; $0:\frac{3}{5}=0$.

2. The quotient resulting from the division of zero by zero is indeterminate. Here, any number satisfies the definition of a quotient (see Sec. 23, Item 4). For example, we could set $0:0=5$ because $5\cdot 0=0$, but we could also put $0:0=3\frac{5}{7}$ because $3\frac{5}{7}\cdot 0=0$. Thus, the problem of dividing zero by zero has infinitely many solutions and is meaningless without further information, which has to indicate how the dividend and divisor varied before they became zero. If this is known, then in most cases it is possible to give meaning to the expression 0:0. For instance, if we know that the dividend took on the successive values $\frac{3}{100}$, $\frac{3}{1000}$, $\frac{3}{10{,}000}$, etc., and the divisor assumed the values $\frac{7}{100}$, $\frac{7}{1000}$, etc., then the quotient, meantime, assumed the values $\frac{3}{100}:\frac{7}{100}=\frac{3}{7}$; $\frac{3}{1000}:\frac{7}{1000}=\frac{3}{7}$, etc., which is to say, remained equal to $\frac{3}{7}$, and so the quotient of 0:0 can, here, be taken equal to $\frac{3}{7}$.

In such cases, one speaks of evaluating the indeterminate expression 0:0 (see Sec. 217, Example 2). Higher mathematics offers a number of techniques for evaluating the indeterminate expression 0:0, but in certain cases the tools of elementary mathematics suffice.

3. The quotient obtained by dividing some nonzero number by zero does not exist because in this case no number can satisfy the definition of a quotient (see Sec. 23, Item 4).

As an example, take 7:0. No matter what number we take to test this out (say, 2, 3, 7) we get the same unsatisfactory answer ($2\cdot 0=0$, $3\cdot 0=0$, $7\cdot 0=0$) whereas what we need is 7. We can say that the problem of dividing a nonzero number by zero has no solution.

On the other hand, a nonzero number may be divided by a number arbitrarily close to zero, and the closer the divisor is to zero, the greater will be the quotient. Thus, if we divide $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$ $\frac{1}{10{,}000}$, etc., by 7, we obtain the quotients 70, 700, 7000, 70,000, etc., which increase without bound. For this reason, we say that the quotient obtained by dividing 7 by 0 is "infinitely great" or is "equal to infinity", and we write: $7:0=\infty$. The meaning of this expression is that if the divisor approaches zero and the dividend remains equal to 7 (or approaches 7), then the quotient will increase without bound.

38. The Whole and a Part

1. Finding a part from the whole. To find some part of a number, multiply it by the fraction expressing that part.

Example. A committee of 120 members meets for elections. Two thirds of the body must be present. How many members must be present for the meeting to take place?

Solution: $120\cdot\frac{2}{3}=80$.

2. Finding the whole from a part. To find a number when one of its parts is known, divide the number by the fraction expressing the given part.

Example. The dead weight of an ox is 3/5 the live weight. What is the live weight of an ox whose dead weight is found to be 420 kg?

Solution: $420:\frac{3}{5}=700$ (kg).

3. Expressing a part as a fraction of the whole. To express a part as a fraction of the whole, divide by the whole number.

Example. Four students are absent in a class of 30. What part of the class is absent?

Solution: $4:30=\frac{4}{30}=\frac{2}{15}$.

39. Decimal Fractions

Computations involving common fractions become very unwieldy if the denominators are big numbers. The main difficulty lies in reducing the fraction to a common denominator. This is because there is no system in the choice of denominator, any number will do. That is why even in antiquity the idea arose of choosing regularly (not arbitrarily) certain parts of a unit (in common fractions they play the part of denominators). The most ancient systematic fractions which were used in Babylonia some 4000 years B.C. and were passed on by the ancient Greek astronomers to the astronomers of Western Europe were the sexagesimal fractions (see Sec. 21, Item 4). At the end of the 16th century when intricate computations involving fractions were widely used in all spheres of life, systematic fractions of a different kind came into use: decimal fractions (see Sec. 45). Here, the unit is divided into ten equal parts (tenths), each tenth into tenths (hundredths), and so on. The advantage of decimal fractions over other systematic fractions lies in the fact that they are based on the same number scale (of ten) as are the whole numbers. As a result, both notation and rules of handling decimals are essentially the same as those used when dealing with whole numbers (integers).

When writing decimals there is no need to indicate the name of the parts (denominator); it is clear from the position of the appropriate digit. We first write the integral part, then a dot (decimal point),* then the first digit after the decimal point represents tenths, the second digit, hundredths, the third, thousandths, etc. The digits which come after the decimal point on the right are termed decimals (decimal places).

Example. 7.305 signifies seven units, 3 tenths, 5 thousandths (the zero indicates that there are no hundredths), or $7.305=7+\frac{3}{10}+\frac{0}{100}+\frac{5}{1000}$.

One of the advantages of decimal fractions is that the expression of the fractional part is given directly in a form reduced to a common denominator:

$$7.305=7\,\frac{305}{1000}$$

the number following the decimal point (305) is the numerator of the fractional part and the denominator of the fraction is the number which shows how many parts are indicated by the last decimal place (in our case it is 1000).

If a decimal fraction does not have an integral part, then the best usage recommends a zero to be placed before the decimal point; for example: $\frac{35}{100}=0.35$.

---

* In some countries, a comma is used to separate the whole number from the decimal fraction.

40. Properties of Decimal Fractions

1. Annexing zeros to the right of a decimal fraction does not change the value.

Example. $12.7=12.70=12.700$, etc. (for the difference that is made between 12.7 and 12.70 see Sec. 48).

2. Dropping the zeros at the end of a decimal fraction does not change the value of the fraction.

Example. $0.00830=0.0083$. (Zeros that do not come at the end of the fraction cannot be omitted).

3. A decimal fraction is increased (multiplied by) 10, 100, 1000, etc. times if the decimal point is moved one place, two places, three places, etc. to the right.

Example. The number 13.058 becomes 100 times larger if we write 1305.8.

4. A decimal fraction is reduced dy 10, 100, 1000, etc. times if the decimal point is moved leftwards one, two, three, etc. places.

Example. 176.24 decreases 10 times if we write 17.624; it will be 1000 times less if we write 0.17624.

These properties enable us to perform rapid multiplication and division by the numbers 10, 100, 1000, etc.

Examples. $12.08\cdot 100=1208$, $12.08\cdot 10{,}000=120{,}800$ (first write 12.08 as 12.0800 and then move the decimal point to the right four places); $42.03:10=4.203$, $42.03:1000=0.04203$ (first write 42.03 as 0042.03 and then move the decimal point three places to the left).

41. Addition, Subtraction and Multiplication of Decimal Fractions

Addition and subtraction of decimal fractions are performed in the same way as the addition and subtraction of whole numbers; take care to write each digit in its proper place (tenths under tenths, hundredths under hundredths, and the like).

Work:

$$\begin{array}{r} 2.3 \\ +\ 0.02 \\ 14.96 \\ \hline 17.28 \end{array}$$

Example. $2.3+0.02+14.96=17.28$.

Multiplication of decimal fractions. Multiply the given numbers as whole numbers disregarding the decimal point. Then insert the decimal point using the following rule: *take the sum of the decimal places in all factors and point off that number of decimal places in the product.*

Example 1. $2.064 \cdot 0.05$. Multiply the whole numbers $2064 \cdot 5 = 10{,}320$. The first factor had three decimal places, the second two, and so the product must have five places set off after the decimal point. This yields 0.10320. The zero at the end of the fraction is dropped and we have $2.064 \cdot 0.05 = 0.1032$.

In this method, do not drop zeros before pointing off the decimal places (when multiplying by the method given in Sec. 55, the zeros may be dropped).

Example 2. $1.125 \cdot 0.08$; $1125 \cdot 8 = 9000$. The number of decimal places is $3 + 2 = 5$. Annexing zeros to the left of 9000 (009000) we point off five decimal places to get $0.09000 = 0.09$.

42. Division of a Decimal Fraction by an Integer

1. If the dividend is less than the divisor, write a zero in the integral part of the quotient and then the decimal point. Then, disregarding the decimal point, annex to the integral part of the dividend the first digit of its fractional part; if the resulting number is less than the divisor, put a zero after the decimal point and annex another digit of the dividend; if we still have a number less than the divisor, put down another zero, etc., until we get a number exceeding the divisor. The division is then performed in the same way as for whole numbers (integers). Note that the dividend may be "expanded" without bound rightwards from the decimal point by adding zeros.

Note. The process of division, as described above, may never end. In that case, the quotient cannot be expressed exactly by means of a decimal fraction. But we can terminate the process at any point and obtain an approximate result (see Sec. 44 below).

Example 1. 13.28:64.

Work:

```
13.28 |  64
12.8  |0.2075
-----
  48
  480
  448
  ---
   320
```

Here the number 132 exceeds the divisor as soon as we shift the decimal point one place to the right, and so there is no zero directly after the decimal point. But after we bring down the next digit the first remainder (48) is less than the divisor, so we put a zero in the dividend (expanded, the dividend becomes 13.280). This zero is then brought down and we can continue the division process. We get a remainder of 32 and again have to bring down a zero (the dividend then becomes 13.2800).

Example 2. 0.48:75.

Work:

$$\begin{array}{r|l} 0.480 & \;75 \\ \cline{2-2} \underline{450} & 0.0064 \\ \underline{300} & \end{array}$$

Here, moving the decimal point one place to the right we get 4, which is less than 75; write zero in the quotient after the decimal point; move the decimal point one more place to the right in the dividend to get 48, which is still less than 75. Put a second zero in the quotient to the right of the decimal point. Adding one zero to the fraction we get 0.480, etc.

2. If the dividend is larger than the divisor, first divide the integral part; write down the result in the quotient and place the decimal point. The division process then proceeds as in the preceding case.

Example 3. 542.8:16.

Work:

$$\begin{array}{r|l} 542.8 & \;16 \\ \cline{2-2} \underline{48} & 33.925 \\ 62 & \\ \underline{48} & \\ 148 & \\ \underline{144} & \\ 40 & \\ \underline{32} & \\ \underline{80} & \end{array}$$

Dividing the integral part, we get a quotient of 33 and a remainder (the second remainder) of 14. Put the decimal point after 33 and bring down the next digit, 8. Divide 16 into 148 to get 9, which is the first digit after the decimal point, etc.

The same procedure is used in dividing a whole number by a whole number if it is desired to give the quotient as a decimal fraction.

Example 4. 417:15.

Work:

$$\begin{array}{r|l} 417 & \;15 \\ \cline{2-2} \underline{30} & 27.8 \\ 117 & \\ \underline{105} & \\ \underline{120} & \end{array}$$

Here, the decimal point is inserted after the last integral remainder (12) is obtained. The dividend 417 may be written as 417.0; it is then represented as a decimal.

Alternative method of division.
The process of division can be written differently:

Example 5: Divide 45, 837 by. 312

```
        14691
    ----------
312| 45837
     312
     ----
     1463
     1248
     ----
      2157
      1872
      ----
      2850
      2808
      ----
        420
        312
        ---
        108 (remainder)
```

43. Division Involving Decimal Fractions

To divide a decimal fraction (or a whole number) by a decimal fraction, drop the decimal point in the divisor and move the decimal point rightwards in the dividend the same number of places as in the fractional part of the divisor (if necessary, annex zeros at the end of the dividend). Division can now be performed as indicated in the preceding section.

Example. 0.04569:0.0012.

Work:

```
456.9  |   12
36     |--------
-------| 38.075
 96
 96
 --
  90
  84
  --
  60
  --
```

There are four decimal places in the fractional part of the divisor, and so we move the decimal point four places to the right in the dividend to get 456.9. Now divide 456.9 by 12.

44. Changing a Decimal Fraction to a Common Fraction and Vice Versa

To change a decimal fraction to a common fraction, drop the decimal point and make the resulting number the numerator of the fraction; the denominator is the number indicated by the last decimal place. It is desirable, if possible, to reduce the fraction to lowest terms.

If the decimal exceeds unity, it is best to change only the decimal part to a common fraction and leave the integral part unchanged.

Example 1. Change 0.0125 to a common fraction. The last decimal, 5, is in the ten-thousandths place, and so the denominator will be 10,000 We have $0.0125=\frac{125}{10{,}000}=\frac{1}{80}$.

Example 2. $2.75=2\frac{75}{100}=2\frac{3}{4}$, or $2.75=\frac{275}{100}=\frac{11}{4}$. The former procedure is to be preferred; that is, leave unchanged the 2 to the left of the decimal point and change 0.75 to a common fraction.

In order to change a common fraction to a decimal, divide the numerator by the denominator using the rule in Sec. 42 (see Example 4).

Example 3. Change the fraction $\frac{7}{8}$ to a decimal. Divide 7 by 8 to get 0.875.

In most cases the division process goes on without end. Then the common fraction cannot be changed into a decimal fraction exactly, which is actually never required in practice. The division is terminated when the quotient has as many decimal places as required in a given practical situation.

Example 4. It is required to divide 1 kilogram of coffee into three parts. The weight of each is 1/3 kg. To weigh this quantity, we have to express it in tenths of a kilogram (since there are no weights of one-third of a kilogram). Dividing 1 by 3, we get: $1:3=0.333\ \ldots$. The division can be continued endlessly with new threes appearing in the quotient. But small weights (say, less than 1 gram) are not indicated by ordinary scales; what is more, the coffee beans themselves weigh more than a gram each. Only hundredths of a kilogram (10 grams) are of practical interest in this case. And so we take $\frac{1}{3}$ kg ≈ 0.33 kg.

For greater accuracy, it is accepted usage to make allowance for the value of the last rejected digit. If it exceeds 5, the retained digit is increased by unity.

Note. Even when a common fraction can be expressed exactly as a decimal, this is not done in most cases. The division process is terminated as soon as the required degree of accuracy is attained.

Example 5. Change the fraction $\frac{7}{32}$ to a decimal. The exact value is 0.21875. Depending on the accuracy required, the division process is terminated with the second, third, etc. digit of the quotient, and we take $\frac{7}{32} \approx 0.22$, $\frac{7}{32} \approx 0.219$, and so on.

45. Historical Survey of Fractions

The notion of a fraction could develop only after definite conceptions concerning whole numbers had been firmly established. Like the concept of an integer, the concept of a fraction developed gradually. The idea of "one half" * originated much before that of thirds or fourths, and the latter two appeared much earlier than fractions with other denominators. The first notion of a whole number evolved out of the process of counting, the first conception of fractions, out of the process of measuring (lengths, areas, weights, and so on). Many languages have traces of the historical connection between fractions and the existing system of measures. For example, in the Babylonian system of measures and money, 1 talent is composed of 60 minas, one mina making 60 shekels. Accordingly, Babylonian mathematics made extensive use of sexagesimal fractions (see Sec. 21). In the weight and monetary system of ancient Rome, 1 as consisted of 12 ounces (uncia); the Romans accordingly made use of duodecimal fractions. The fraction we call $\frac{1}{12}$ was called an "uncia" by the Romans even when it was used for measuring lengths or other quantities. The Romans called $\frac{1}{8}$ one and a half ounces, and so forth.

Our common fractions were widely used by the ancient Greeks and Hindus. The rules for handling fractions given by the Hindu scholar Brahmagupta (8th century) differ but slightly from our own rules. Our way of writing fractions coincides with the Hindu custom. True, the Hindus did not use a fraction bar. The Greeks wrote the denominator above the numerator, although other forms of notation were used more often. For example, they wrote (using other symbols, naturally) 35^{x} (three fifths).

The Hindu symbolism for fractions and rules for handling fractions spread into the Muslim world in the 9th century

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* In all languages, the concept "half" has a special name not connected with the word "two". Originally "half" meant one of two parts (which were not necessarily equal).

due to al-Khowarizmi (see Sec. 21), and thence to Western Europe in the 13th century thanks to the Italian merchant and scholar Leonardo of Pisa (also known as Fibonacci).

Besides common fractions, sexagesimal fractions were in use, especially in astronomy. The latter subsequently gave way to decimal fractions, which were first introduced by the celebrated Samarkand scholar al-Kashi (14th to 15th century). In Europe, decimal fractions were introduced by the Flemish mathematician and engineer (he was also a merchant) Simon Stevin (1548—1620).

46. Percentage

The expression "per cent" (from the Latin "per centum", "by the hundred") means a hundredth part. Symbolically, 1% stands for 0.01; 27% for 0.27; 100% for 1; 150% for 1.5, etc. (the symbol for percentage, %, is a distortion of the notation c_{to}, which is a contraction of the word "cento").

1% of a sum means 0.01 of it; to fulfil a plan means to complete 100% of it; whereas fulfillment by 150% would mean that one and a half quotas of the planned amount had been completed, and so forth.

To find the percentage expression of a given number, multiply the number by 100 (or, what is the same, move the decimal point two places to the right).

Examples. Expressed as a percentage, 2 is 200%; the number 0.357 is 35.7%, the number 1.753 is 175.3%.

To change a percent to a number, divide the percent by 100 (or, what is the same, move the decimal point two places to the left).

Examples. $13.5\% = 0.135$, $2.3\% = 0.023$, $145\% = 1.45$, ${}^2/_5\% = 0.4\% = 0.004$.

The three principal problems involving percentage are:

Problem 1. Find the indicated percent of a given number. (cf. Sec. 38, Rule 1). Multiply the number by the percent and divide by 100 (or, what is the same, move the decimal point two places to the left: in other words, the given number is multiplied by the fraction expressing the given percent).

Example. A planned quota in coal production is 2860 tons per day. A mine pledges to do 115% of the plan. How many tons of coal will it mine per day?

Solution. (1) $2860 \cdot 115 = 328{,}900$.

(2) $328{,}900 : 100 = 3289$ tons

(which is equivalent to $2860 \cdot 1.15 = 3289$).

Problem 2. Find a number on the basis of a given percent (cf. Sec. 38, Rule 2). The given quantity is divided by the percent and then multiplied by 100 (or the decimal point is moved two places to the right, which is to say the given number is divided by the fraction expressing the given percent).

Example. In processing sugar beets, 12.5% of the weight of the beets is granulated sugar. What quantity of beets has to be processed to produce 3000 centners of granulated sugar?

Solution. (1) $3000:12.5=240$. (2) $240\cdot 100=24{,}000$ (centners) (which is tantamount to writing $3000:0.125=24{,}000$).

Problem 3. Find what percent one number is of another (cf. Sec. 38, Rule 3). Multiply the first number by 100 and divide by the second number.

Example 1. A new burning process for brick manufacture made it possible to increase the output of bricks per cubic metre of furnace from 1200 to 2300 bricks. What was the increase in brick output in percentage?

Solution.

(1) $2300-1200=1100$,
(2) $1100\cdot 100=110{,}000$,
(3) $110{,}000:1200\approx 91.67$.

Brick output increased by 91.67%.

Example 2. According to the seven-year plan, the petroleum output in the USSR was to reach 161 million tons in 1961. Actually, 166 million tons were produced. Give the fulfillment of the 1961 plan in percentage.

Solution.

(1) $166\cdot 100=16{,}600$,
(2) $16{,}600:161\approx 103.1$.

Petroleum output in 1961 was 103.1% of the planned amount.

Note 1. In all three types of problems, the sequence of operations can be changed (say, in the last problem, we could first divide and then multiply by 100).

Note 2. The example which follows is to serve as a warning against a mistake that is very frequently made.

It is required to find out the price of a metre of cloth prior to a price reduction if after a price reduction of 15% the price is 12 roubles per metre. Sometimes, 15% of 12 roubles is found, that is, $12\cdot 0.15=1.8$. This is followed by the addition $12+1.8=13.8$, and it is taken that the old price was 13.8 roubles per metre. This is not so because the percent of reduction is established with respect to the earlier prices, and 1.8 roubles is not 15% of 13.8 roubles but about 13% (see Problem 3). The correct solution is this: after the price reduction, the cloth cost 100% — 15% = 85% of the earlier price. And so the old price (see Problem 2) was 12:0.85 = 14.12 roubles per metre.

Note 3. In working percentage problems it is best to take advantage of the methods of approximate computations (see the sections which follow).

47. Approximate Calculations

The numbers we deal with in everyday affairs are of two kinds. Some state exact magnitudes, others give only approximate values. Thus, we have exact numbers and approximate numbers. We often take an approximate value in place of an exact value simply because the latter is not required. In many cases it is simply impossible to find a number exactly.

Example 1. The number of pages in a book is exact. This book has 423 pages.

Example 2. A hexagon has 9 diagonals, which is an exact number.

Example 3. A salesman weighs 50 grams of butter. 50 is an approximate number because the scales are not sensitive to an increase or decrease of 0.5 gram.

Example 4. The distance by railway from Moscow to Leningrad is 651 kilometres. The number 651 is an approximate number because our measuring instruments are not exact.

Operations involving approximate values yield approximate values. What is more, inexact digits may result from operations on the exact digits of the given numbers.

Example 5. In multiplying the approximate numbers 60.2 and 80.1 let us suppose that all the indicated digits are correct so that the true values can differ from the approximate ones only in hundredths, thousandths, etc. The product is 4822.02. Here, not only the hundredths and tenths digits but even the units digits may be incorrect. For example, suppose that the factors were obtained by rounding off (see Sec. 49) the exact numbers 60.25 and 80.14. Then the exact product is 4828.435, so that the units digit in the approximate product (2) differs from the exact figure (8) by 6 units.

If we know the degree of accuracy of the starting figures, the theory of approximate computations permits us (1) to estimate the degree of accuracy of the results prior to performing the operations, (2) to take the initial numbers with the desired degree of accuracy so as to ensure the required accuracy of the result without involving the computor in extra needless computations, (3) to rationalize the very process of computation by omitting computations that do not affect the final exact figures of the result.

48. Notation of Approximate Numbers

In approximate calculations we distinguish between 2.4 and 2.40, between 0.02 and 0.0200 and so on. The notation 2.4 means that only the units and tenths are correct; the true value of the number may be, say, 2.43 or 2.38 (when the digit 8 is dropped, we round the preceding digit upwards; see Sec. 49). The notation 2.40 means that the hundredths are correct; the true number may be 2.403 or 2.398 but not 2.421 or 2.382.

This distinction also holds true for whole numbers. The notation 382 means that all digits are correct; now if there is any doubt about the last digit, the number is rounded off and is written as $38 \cdot 10$ and not 380. If we write 380 this means that the last digit (0) is true. If in the number 4720 only the first two digits are correct, then it must be written as $47 \cdot 10^2$; this number can also be written in the form $4.7 \cdot 10^3$, etc.

The *significant digits* of a number are all the correct digits (except for zeros) which stand at the beginning of the number. For example, in 0.00385 there are three significant digits; in the number 0.03085 there are four significant digits; in 2500 there are four, and in $2.5 \cdot 10^3$ there are two.

49. Rules for Rounding Off Numbers

In approximate computations it is frequently necessary to round off numbers (both approximate and exact), which means dropping one or more of the last digits. To ensure that the rounded number is as close as possible to the original number, use the following rules:

Rule 1. If the first of the discarded digits exceeds 5, then the last digit kept is increased by unity. The increase is also made when the first digit kept is equal to 5 and is followed by one or more significant digits (for the case when the discarded 5 is not followed by any digits, see Rule 3 below).

Example 1. Rounding the number 27.874 to three significant digits, we write 27.9. The third digit 8 is increased to 9 since the first discarded digit, 7, exceeds 5. The number 27.9 is closer to the original number than the rounded (but not increased) number 27.8.

Example 2. Rounding the number 36.251 to the first decimal, we write 36.3. The tenths digit 2 is increased to 3 because the first discarded digit is equal to 5 and it is followed by one significant digit, 1. The number 36.3 is closer to the initial number (though only slightly) than the unincreased number 36.2.

Rule 2. If the first digit dropped is less than 5, no increase is made.

Example 3. Rounding 27.48 to the nearest whole number, we write 27. This number is closer to the given one than 28.

Rule 3. If the digit 5 is dropped, and no significant digits come after it, the rounding is done to the closest even number; that is, the last retained digit is left unchanged if it is even and is increased if it is odd. The reason for this rule is given below (see note).

Example 4. Rounding 0.0465 to the third decimal place, we write 0.046. We do not increase the last digit kept since it is even. The number 0.046 is just as close to the given one as is 0.047.

Example 5. Rounding 0.935 to the second decimal place, we write 0.94. The last retained digit 3 is increased because it is odd.

Example 6. Rounding the numbers

6.527, 0.456, 2.195, 1.450, 0.950, 4.851, 0.850, 0.05 to the first decimal place, we get

6 5, 0.5, 2.2, 1.4, 1.0, 4.9, 0.8, 0.0

Note. When applying Rule 3 to the rounding off of one number do not increase the accuracy of the rounding process (see Examples 4 and 5). However, if the process is performed repeatedly, there will be roughly just as many numbers with excess as with deficit. The mutual compensation of errors will ensure the highest possible accuracy of the result.

Rule 3 can be modified and made to apply to rounding off to the closest odd number. The accuracy will be the same but even digits are more convenient than odd digits.

50. Absolute and Relative Errors

The *absolute error* (or, simply, error) of an approximate number is the difference between the number and its exact value (the small number is subtracted from the greater).*

Example 1. There are 1284 employees in a given institution. Rounding this number off to 1300, we get an absolute error of $1300-1284=16$. Rounding off to 1280, we have an absolute error of $1284-1280=4$.

The *relative error* of an approximate number is the ratio (see Sec. 62) of the absolute error of the approximate number to the number itself.

Example 2. A school has a student body of 197. Rounding this number to 200, we obtain an absolute error of $200-197=3$. The relative error is equal to $\frac{3}{197}$ or, rounded, $\frac{3}{200}=1.5\%$.

In most cases it is impossible to determine the exact value of an approximate number and, hence, the exact value of the error. However, it is almost always possible to establish that the error (absolute or relative) does not exceed a certain number.

Example 3. A salesman weighs a watermelon on pan scales. The smallest of the set of weights is 50 grams. The result is 3600 grams. This is an approximate number. The exact weight of the watermelon is not known. However, the absolute error does not exceed 50 grams, and the relative error does not exceed $\frac{50}{3600}\approx 1.4\%$.

A number which definitely exceeds the absolute error (or, at worst, is equal to it) is called the *limiting absolute error*. A number which definitely exceeds the relative error (or, at worst, is equal to it) is called the *limiting relative error*.

In Example 3, we can take 50 grams for the limiting absolute error and 1.4% for the limiting relative error.

The magnitude of a limiting error is not quite definite. Thus, in Example 3 we can take 100, 150 and generally any number over 50 grams for the limiting absolute error. In practical cases the smallest possible value of the limiting error is taken. When the exact magnitude of the error is known, this serves, at the same time, as the limiting error.

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* In other words, if a is an approximate number and x its exact value, then the absolute error is the absolute value (Sec. 69) of the difference $a-x$. In some manuals, the absolute error is defined as the difference itself $a-x$ (or the difference $x-a$). This quantity can be positive or negative.

For every approximate number, we must know its limiting error (absolute or relative). When it is not directly indicated, it is assumed that the limiting absolute error represents one half of one unit of the last written digit. For instance, if we have the approximate number 4.78 without the limiting error indicated, then the limiting absolute error is assumed to be 0.005. With this convention, we can always do without indicating the limiting error of a number rounded off by the rules of Sec. 49.

The limiting absolute error is denoted by the Greek letter delta (Δ); the limiting relative error, by the lower-case Greek delta (δ). If an approximate number is denoted by a, then $\delta=\frac{\Delta}{a}$.

Example 4. A pencil is measured with a ruler calibrated in millimetres and yields a result of 17.9 cm. What is the limiting relative error of this measurement?

Here, $a=17.9$ cm; we can take Δ to be equal to 0.1 cm, since it is not difficult to measure a pencil to within 1 mm, yet it will not be possible to reduce substantially the limiting error (with practice, it is possible to read 0.02 cm, and even 0.01 cm, on a good ruler but at the very edge of the ruler the discrepancy may be greater). The relative error is $\frac{0.1}{17.9}$. Rounding off, we get $\delta=\frac{0.1}{18}\approx 0.6\%$.

Example 5. A cylindrical piston is about 35 mm in diameter. To what degree of accuracy must a measurement be made with a micrometer so that the limiting relative error is 0.05%?

Solution. It is given that the limiting relative error must constitute 0.05% of 35 mm. Consequently (Sec. 46, Problem 1), the limiting absolute error is $\frac{35\cdot 0.05}{100}=0.0175$ (mm) or, rounding upwards, 0.02 (mm).

We can use formula $\delta=\frac{\Delta}{a}$. Substituting $a=35$, $\delta=0.0005$, we get $0.0005=\frac{\Delta}{35}$. Thus,

$$\Delta=35\cdot 0.0005=0.0175 \text{ (mm)}$$

51. Preliminary Rounding Off in Addition and Subtraction

If the given numbers do not all end in the same digit place (order), round off before performing the addition or subtraction. In other words, retain only those digit places that are good for all addends, the others being dropped as useless. For a small number of addends, all digits of the sum, except the last, will be correct. The last digit may not be quite exact. This inexactness can be reduced to a minimum if we take into account the digits of the next digit place *(extra digits)*.

Example 1. Find the sum of $25.3+0.442+2.741$.

Without rounding off the terms, we get 28.483. The last two digits are useless since there is a possible inaccuracy of several hundredths in the first addend. Rounding the sum to exact digits (that is, to tenths), we get 28.5. If we first round off to exact digits, then we readily get $25.3+0.4+2.7=28.4$. The tenths digit is less by 1. Taking the hundredths digits as well, we get $25.3+0.44+2.74=28.48$, which, rounded, is 28.5. The digit 5 is more reliable than 4, though it might very well be that the true figure is precisely 4.*

When using extra digits, arrange the computation as indicated in the accompanying scheme with the extra digits separated by a vertical line:

$$\begin{array}{lr|l} \textbf{Work:} & 25.3 & \\ & 0.4 & 4 \\ + & 2.7 & 4 \\ \cline{2-2} & 28.5 & \end{array}$$

Example 2. Find the sum of $52.861+0.2563+8.1+57.35+0.0087$.

Without using any extra digits (we retain only rounded tenths; see rules for rounding, Sec. 49), we get 118.7. With extra digits, we have 118.6. In the latter result, the tenths digit may prove to be incorrect due to the inaccuracy of the third addend; a 5 may appear in place of 6 (if the third addend has been rounded off from 8.06). But 6 is much more reliable. At any rate 7 cannot be correct. Extra digits yield an improvement, but only a slight one. Compare the two schemes: on the left without extra digits, on the right using extra digits:

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* If we assume the first addend to be 25.26 rounded, then the sum to hundredths would be 28.44, or approximately 28.4. However, if 25.3 is the rounded number 25.27 or 25.28, etc., then the sum will be 28.5 after rounding.

Work:
$$\begin{array}{r} 52.9\\ +\ \ 0.3\\ 8.1\\ 57.4\\ \hline 118.7 \end{array} \qquad \begin{array}{r|l} 52.8 & 6\\ 0.2 & 6\\ 8.1 & \\ +\ 57.3 & 5\\ 0.0 & 1\\ \hline 118.6 & \end{array}$$

52. Error of a Sum and a Difference

The limiting absolute error of a sum is equal to the sum of the limiting absolute errors of the separate addends.

Example 1. The approximate numbers 265 and 32 are added. Let the limiting error of the former be 5, of the latter, 1. Then the limiting error of the sum is equal to $5+1=6$. Thus, if the true value of the former number is 270, of the latter, 33, then the approximate sum $(265+32 = 297)$ is 6 less than the true value $(270+33=303)$.

Example 2. Find the sum of the approximate numbers $0.0909+0.0833+0.0769+0.0714+0.0667+0.0625+0.0588+0.0556+0.0526$.

Addition yields 0.6187. The limiting error of each addend is 0.00005; the limiting error of the sum is $0.00005\times 9=0.00045$. Thus, there is a possible error of up to 5 units in the last (fourth) decimal place of the sum, and so we round off the sum to the third decimal place (thousandths). This yields 0.619; where all the decimal places are correct.

Note. A large number of addends usually makes for a balancing of errors; for this reason, only in exceptional cases does the true error of a sum coincide with the limiting error or come close to it. That these cases are rare is seen from Example 2 where we had 9 addends. The true value of each of them can differ in the fifth decimal place from the given approximate value by 1, 2, 3, 4 or even 5 units either way. For example, the first addend may be greater than its true value by 4 units of the fifth decimal place, the second by two units, the third, less by one unit, etc. Calculations show that the number of all possible cases of the distribution of errors is about 1000 million. But only in two cases can the error of the sum reach the limiting error of 0.00045. This occurs when (1) the true value of each addend exceeds the approximate value by 0.00005, and (2) the true value of each addend is less than the approximate value by 0.00005. Thus, the cases when the error of a sum coincides with the limiting error constitute only 0.0000002% of all possible cases.

Further calculations show that cases when the error of a sum of nine addends can exceed three units of the last decimal place are also very rare. They amount to only 0.07% of all possible cases. An error can exceed two units of the last decimal place in 2% of all possible cases, one unit, in roughly 25%. In the remaining 75% of the cases, the error of nine addends does not exceed one unit of the last decimal place.

Example 3. Assuming the addends of Example 2 to be exact numbers, * let us round them off to thousandths and add. The limiting error of the sum will be $9 \cdot 0.0005 = 0.0045$. Yet we have

$$0.091 + 0.083 + 0.077 + 0.071 + 0.067 + 0.062 + 0.059 \\ + 0.056 + 0.053 = 0.619$$

That is to say, the approximate sum differs from the true sum by 0.0003, which is a third of a unit of the last decimal place of the approximate numbers. All three decimals of the approximate sum are correct, although theoretically the last decimal might be glaringly inexact.

Let us round off to hundredths in our addends. Now the limiting error of the sum is $9 \cdot 0.005 = 0.045$. Yet we get $0.09 + 0.08 + 0.08 + 0.07 + 0.07 + 0.06 + 0.06 + 0.06 + 0.05 = 0.62$. The true error comes out to only 0.0013, which is $\frac{1}{8}$ of a unit of the last decimal of the approximate numbers.

The limiting absolute error of a difference is equal to the sum of the limiting absolute errors of the minuend and the subtrahend.

Example 4. Let the limiting error of the approximate minuend 85 be 2, and the limiting error of the subtrahend 32 be 3. The limiting error of the difference $85 - 32 = 53$ is $2 + 3 = 5$. Indeed, the true values of the minuend and subtrahend may be equal to $85 + 2 = 87$ and $32 - 3 = 29$. Then the true difference is $87 - 29 = 58$. It differs from the approximate difference 53 by 5.

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* These addends are obtained by changing the common fractions $\frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \ldots, \frac{1}{19}$ to decimals to within the fourth decimal place. The reader can take other numbers at random.

The limiting relative error of a sum and a difference can easily be found by first computing the limiting absolute error (see Sec. 50).

The limiting relative error of a sum (but not of a difference) lies between the smallest and largest of the relative errors of the addends. If all the addends have the same (or roughly the same) limiting relative error, then the sum also has the same (or roughly the same) limiting relative error. In other words, in this case the accuracy of the sum (expressed as a percentage) is not inferior to the accuracy of the addends. For a large number of addends, the sum (as a rule) is much more accurate than the addends (for the reason explained in the note of Example 2).

Example 5. In each addend of the sum $24.4+25.2+24.7=74.3$ the limiting relative error is approximately the same, namely, $0.05:25=0.2\%$. It is the same for the sum as well. Here, the limiting absolute error is equal to 0.15, the relative error $0.15:74.3 \approx 0.15:75=0.2\%$.

In contradistinction to the sum, the difference between two approximate numbers may be less exact than the minuend and subtrahend. The "loss of accuracy" is particularly great when the minuend and subtrahend differ only slightly.

Example 6. Measurements of the outer and inner diameters of a thin-walled pipe yielded 28.7 mm and 28.3 mm, respectively. Using these figures, we find the wall thickness: $\frac{1}{2}\cdot(28.7-28.3)=0.2$ (mm). The limiting relative error of the minuend (28.7) and the subtrahend (28.3) is the same: $\delta=0.2\%$. The limiting relative error of the difference 0.4 (and also of half the difference, 0.2) comes out to 25%.

If follows, from the foregoing, that whenever possible one should avoid computing a desired quantity by subtracting nearly equal numbers. Cf. Sec. 90, Example 9.

53. Errors in a Product

The *limiting relative error of a product* is approximately equal to the sum of the limiting relative errors of the factors. (For the exact value of the limiting error see note of Example 1).

Example 1. Two approximate numbers, 50 and 20, are multiplied together. Let the limiting relative error of the first factor be 0.4%, of the second factor, 0.5%. Then the limiting relative error of the product $50\times20=1000$ is approximately 0,9%. Indeed, the limiting absolute error of the first factor is $50 \cdot 0.004 = 0.2$, of the second, $20 \cdot 0.005 = 0.1$. Therefore, the true value of the product does not exceed $(50+0.2) \cdot (20+0.1) = 1009.02$, and is not less than $(50-0.2) \times (20-0.1) = 991.02$. If the true value of the product is 1009.02, then the error of the product is equal to $1009.02 - 1000 = 9.02$ and if it is 991.02, then the error of the product is $1000 - 991.02 = 8.98$. These two cases are the most unfavourable ones. Hence the limiting absolute error of the product is 9.02. The limiting relative error is equal to $9.02 : 1000 = 0.902\%$, which is approximately 0.9%.

Note. Denote the limiting relative error of a product by the letter δ, the limiting relative error of the factors by δ_1 and δ_2 (in Example 1, $\delta_1 = 0.004$, $\delta_2 = 0.005$, $\delta = 0.00902$).

Our rule (for two factors) then looks like this:

$$\delta \approx \delta_1 + \delta_2$$

The exact expression of δ is

$$\delta = \delta_1 + \delta_2 + \delta_1 \delta_2$$

That is, the limiting relative error of a product is always greater than the sum of the limiting relative errors of the factors; it exceeds this sum by the product of the relative errors of the factors. The excess is ordinarily so small that it can be ignored. Taking Example 1, we have $\delta = 0.004 + 0.005 + 0.004 \cdot 0.005 = 0.00902$. The excess here is $0.00902 - 0.009 = 0.00002$, which is about 0.2% of the approximate value of the limiting relative error. This excess is so small that it can be disregarded.

Example 2. Suppose the approximate numbers 53.2 and 25.0 are multiplied together. The limiting absolute error of each is 0.05. Therefore, $\delta_1 = 0.05 : 53.2 = 0.0009$, $\delta_2 = 0.05 : 25.0 = 0.002$. The limiting relative error of the product $53.2 \cdot 25.0 = 1330$ is approximately equal to $0.0009 + 0.0020 = 0.0029$. The quantity $\delta_1 \delta_2 = 0.0009 \cdot 0.002 = 0.0000018$ is so small that it is meaningless to take it into account. The limiting absolute error of the product 1330 is $1330 \cdot 0.0029 \approx 4$, so that the last digit of the product (zero) may be incorrect.

Example 3. Find the volume of a room, given the measurements: length 4.57 m, width 3.37 m, height 3.18 m (the limiting absolute errors are 0.005 m). Multiplying these numbers together we find the volume to be 48.974862 m^3. But only two digits are definitely correct here, the third may already have a slight error. Indeed, the limiting relative errors of the factors are:

$$\delta_1 = 0.005:4.57 \approx 0.0011, \quad \delta_2 = 0.005:3.37 \approx 0.0015,$$
$$\delta_3 = 0.005:3.18 \approx 0.0016$$

The limiting relative error of the product is

$$\delta = 0.0011 + 0.0015 + 0.0016 = 0.0042$$

The limiting absolute error of the product $\Delta \approx 49.0 \cdot 0.0042 \approx 0.21$. Thus the third significant digit of the product is unreliable. Hence we take the volume of the room to be 49.0 m^3.

54. Counting Exact Digits in Multiplication

The error in a product may be estimated more simply (more crudely, true) than by the procedure given in Sec. 53. This estimate is based on the following rule.

Let two approximate numbers be multiplied together and let each have k significant digits. Then the $(k-1)$st digit of the product is definitely correct, while the kth digit may not be quite exact. However, the error of a product does not exceed $5^1/_2$ units of the kth digit and only in exceptional cases is close to this limit. Now if the first digits of the factors in a product yield a number exceeding ten (either taking into account or disregarding the effect of the subsequent digits), the error of the product does not exceed one unit of the kth digit.

Example 1. Let us multiply together the approximate numbers 2.45 and 1.22, each of which has three significant digits. In the product, 2.9890, the first two digits are surely correct. The third digit may not be quite exact. For the given values of the factors, the limiting absolute error of the product (it may be found, as was done in Example 1, Sec. 53) constitutes 1.8 units of the third digit (or 0.0018); as a rule, the true error will be still less. Therefore the third digit should be retained and there is no sense in keeping the fourth digit. Rounding off, we have $2.45 \cdot 1.22 \approx 2.99$.

Example 2. Multiply the approximate numbers 46.5×2.82. In the product, 131.130, the first two digits are definitely correct. Since the first digits of the factors (with account taken of the subsequent digits) yield 13 in the product (the first two digits of the number 131.130), it follows that the error of the product definitely does not exceed unity. In this case, the limiting absolute error of the product is only 0.37;

the true error will, as a rule, be less. So the third digit must be retained. It is advisable to retain the fourth digit (which is not quite exact) as an extra digit only when other operations are to follow.

When multiplying together three, four or more approximate numbers, the limiting error increases proportionally (that is to say, it increases over that given above by $1^1/_2$, two, etc. times). However, in most cases, the true error for a small number of factors remains within the same limits (due to a compensation of errors; cf. Sec. 52).

Practical Advice

1. If approximate numbers with the same number of significant digits are being multiplied together, retain the same number of significant digits in the product. The last retained digit will be in doubt.
2. If some of the factors have more significant digits than others, then, prior to multiplication, round off the longer numbers retaining as many digits as the least exact factor, or one more digit (as an extra digit). There is no sense in keeping any more digits.
3. If it is required that the product of two numbers have a prescribed number of reliable digits, then the number of exact digits in each factor (found via computation or measurement) must be one more. If the number of factors is greater than two but less than ten, then the number of exact digits in each factor must, for complete assurance, be two units more than the required number of exact digits. In most practical situations only one extra digit is quite sufficient.

To verify these conclusions, let us consider an example in which we already know the exact values of the approximate numbers being multiplied.

Example 3. Change the product $\frac{1}{3}\cdot\frac{1}{7}\cdot\frac{1}{11}\cdot\frac{1}{13}=\frac{1}{3003}$ to a decimal. Taking 4 significant digits, we get 0.0003330. Now suppose all we know are the approximate values of the factors (the reader is advised to take any other factors):

$$\frac{1}{3}=0.33333,\quad \frac{1}{7}=0.14286,\quad \frac{1}{11}=0.09091,\quad \frac{1}{13}=0.07692$$

and it is required to find the product to two significant digits. To be on the safe side we have to take all factors to four significant digits, that is, we multiply $0.3333 \cdot 0.1429 \times 0.09091 \cdot 0.07692$

(1) We find $0.3333 \cdot 0.1429 = 0.04762857$.

Retaining four significant digits, we get 0.04763.

(2) Perform the following multiplication:

$$0.04763 \cdot 0.0909 = 0.0043300433$$

(3) Retaining four significant digits (three would have sufficed) and performing the last multiplication, we get

$$0.004330 \cdot 0.07692 = 0.0003331$$

The first two significant digits are definitely correct, so that the desired number is 0.00033. We cannot, beforehand, be certain of the correctness of the third significant digit. It proves correct however. The fourth significant digit is not quite exact, but the error does not exceed unity of the corresponding decimal place.

If we carry our answer to six decimal places, then we cannot beforehand be certain of the fifth digit. Actually, however, even the sixth digit is correct. Namely,

(1) $0.333 \cdot 0.143 = 0.047619$,

(2) $0.0476 \cdot 0.0909 = 0.00432684$,

(3) $0.00433 \cdot 0.0769 = 0.000333$

If we carry the computation to five decimals, then for the product we have 0.00032, i. e., the error is 1.3 units of the fifth decimal place.

55. Short-Cut Multiplication

When applying the rules of multiplication of exact numbers to approximate numbers we waste time and effort in the computation of digits that will be dropped at a later stage. The computational procedure can be made more efficient if we are guided by the following rules.

(1) Start the multiplication with the higher digit places of the multiplier (not the lower digit places); when multiplying the multiplicand by the highest digit place of the multiplier, carry out the multiplication completely.

(2) Before multiplication by the next place of the multiplier, cross-line the last digit in the multiplicand; multiply using a shortened multiplicand, but add to the result the rounded product of the given place of the multiplier by the discarded digit of the multiplicand.

(3) Before multiplying by the third (from the beginning) place of the multiplier, cross-line one more digit of the multiplicand (the second from the end); perform the multiplication by the remaining digits of the multiplier and take into account the effect of the digit that was just discarded and so on.

(4) The resulting products are arranged so that all the lower digit-orders are aligned one under another.

(5) There are special rules for setting the decimal point in the product, but it is a good practical rule to make a rough preliminary guess about the magnitude of the product. To avoid errors, it is advisable to cross-line each digit of the multiplier that has been used.

Example 1. Multiply the approximate numbers 6.7428·23.25. Equalize the number of significant digits: drop 8 in the first factor and replace 2 by 3. Compute by the accompanying scheme in the following sequence.

Work:

$$\begin{array}{r} \times\ 6.743 \\ 23.25 \\ \hline 13486 \\ +\ \ 2023 \\ 135 \\ 34 \\ \hline 156.78 \end{array}$$

(1) Disregard the decimal points and multiply 6743 by 2; write out the result in full: 13486; as usual, begin to multiply with $2 \cdot 3 = 6$ (write the 6 under the lowest digit-orders of the factors).

(2) Cross-line the digit of the multiplier that was used (2) and the last digit of the multiplicand (3); multiply the next digit of the multiplier (3) by the shortened multiplicand 674, noting that the cross-lined digit 3 would have given $3 \cdot 3 = 9$ in the product; and so add 1 to the product (from the very beginning of the multiplication, $3 \cdot 4 = 12$; $12 + 1 = 13$; write 3 and carry the 1). The lowest order of the product (3) is written under the lowest order of the preceding product (6).

(3) Cross out the second digit of the multiplier from the beginning and the second digit of the multiplicand from the end, multiply the third digit of the multiplier (2) by the shortened multiplicand 67, noting that multiplication of this digit of the multiplier by the discarded digit of the multiplicand would have yielded 8, and so we add 1 to the product.

(4) Finally, crossing out 2 in the multiplier and 7 in the multiplicand, multiply 5 by 6, noting first that $5 \cdot 7 = 35$, so that to the product $5 \cdot 6 = 30$ we add 4 (which is better than 3 since we would have had to multiply not only the digit 7 but also the discarded digits that follow it).

(5) Add all the products thus obtained to get 15,678.

In order to set the decimal point, we round the factors crudely taking, say, 6 in place of the first and 20 instead of the second. This gives us a rough product of 120, which means that the integral part of our answer is a three-digit number; this means we must point off the first three digits and take 156.78 and not 15.678 or 1567.8. This answer is correct only to the first four digits. We use the last digit (which may contain an error of up to three units) to round off the result to 156.8.

Example 2. $674.3 \cdot 232.5$. The multiplication is performed as in the preceding example. We get 15,678 and to point off the decimal place we carry out the rough computation $600 \cdot 200 = 120{,}000$, which is a six-digit number. Since the integral part of our answer must contain six digits, and our number, 15,678, contains five, we annex a zero on the right; the decimal point lies outside the figures we obtained, i. e., the result of our multiplication yields the whole number 156,780. Since the last digit (zero) is definitely false, we write the answer as $15{,}678 \cdot 10$ or $1568 \cdot 10^2$ (see Sec. 48).

56. Division of Approximate Numbers

Rule 1. The limiting relative error of a quotient is approximately equal to the sum of the limiting relative errors of the dividend and divisor (cf. Sec. 53).

Example 1. Divide the approximate number 50.0 by the approximate number 20.0. The limiting error of the dividend and divisor is 0.05. Then the limiting relative error of the dividend is $\frac{0.05}{50.0} = 0.1\%$, and the limiting relative error of the divisor is $\frac{0.05}{20.0} = 0.25\%$. The limiting relative error of the quotient of 50.0 by $20.0 = 2.50$ must be approximately $0.1\% + 0.25\% = 0.35\%$.

Indeed, the true value of the quotient does not exceed $(50.0 + 0.05):(20.0 - 0.05) = 2.50877$ and is not less than $(50.0 - 0.05):(20.0 + 0.05) = 2.49127$. If the true value of the quotient is 2.50877, then the absolute error comes out to $2.50877 - 2.50 = 0.00877$. But if the true value is 2.49127, then the absolute error comes to $2.50 - 2.49127 = 0.00873$. The foregoing cases are the most unfavourable. Hence, the limiting relative error is $0.00877:2.50 = 0.00351$, or, approximately, 0.35%.

Note. The exact limiting relative error always exceeds the approximate error as computed by Rule 1. The percent of excess is roughly equal to the limiting relative error of the divisor. In our example, the excess is 0.00001, which is 0.29% of 0.0035, whereas the relative error of the divisor is 0.25%.

Example 2. Find the limiting absolute error of the quotient of 2.81 by 0.571.

Solution. The limiting relative error of the dividend is 0.005:2.81 = 0.2%; of the divisor, 0.0005:0.571 = 0.1%; of the quotient, 0.2% + 0.1% = 0.3%. The limiting absolute error of the quotient is approximately equal to $\frac{2.81}{0.571} \times 0.003 = 0.015$. Hence, we are already uncertain about the third significant digit in the quotient 2.81:0.571 = 4.92.

A simpler but more crude estimate of the accuracy of the quotient is based on a count of the exact digits (cf. Sec. 54). The estimate is as follows.

Rule 2. Suppose the dividend and the divisor each have k significant digits. Then the absolute error of the quotient is at worst close to 1.05 units of the $(k-1)$st place (this value will never be attained).

As we see, the limiting error of a quotient is theoretically twice the limiting error of a product (see Sec. 54). Actually, however, the error of a quotient exceeds by 5 units the kth digit only in exceptional cases (once in a thousand). Therefore, *one should take as many significant digits in the quotient as there are in the dividend and divisor.*

If one of the given numbers (dividend or divisor) has more significant digits than the other, then drop all the superfluous digits or retain only the first one (as an extra digit).

If it is required that the quotient have a prescribed number of correct digits, take one extra significant digit in the dividend and divisor.

57. Short Division

In order to avoid superfluous computations, carry out the division of approximate numbers as follows.

(1) Disregarding the position of the decimal points, obtain the first digit of the quotient in the same manner as for whole numbers. If the significant digits of the dividend form a number that exceeds the significant digits of the divisor (both are regarded as whole numbers), then the first digit of the quotient is multiplied by the entire divisor. Otherwise cross out the last digit in the divisor and multiply by the shortened divisor, but take into account the effect of the

discarded digit. Thus, if we divide 2262 by 7646, the first digit of the quotient 2 (22:7 = 3 with a remainder, but 3 is not suitable and so we take 2). It is then multiplied into 764 and 1 is added to the result (this is the first digit of the product $2 \cdot 6 = 12$). This is done immediately upon multiplication by the last digit of the shortened divisor.

(2) The result of multiplying the first digit of the quotient by the divisor (or by the shortened divisor) is written under the dividend, aligning digit place under digit place. Then we find the remainder.

(3) Instead of bringing down a zero to the remainder, we shorten the divisor by cross-lining the last digit (if the shortening has already been done, then drop the last of the remaining digits). Choosing the second digit of the quotient, multiply it by the shortened divisor taking into account the digit just discarded.

(4) Write the result of the multiplication under the first remainder and align the digit places. We then find the second remainder.

(5) Instead of bringing down a zero we shorten the divisor by one more digit, etc.

6) Having obtained the quotient, we set off the decimal point by a rough estimate.

Example 1. 58.83:9.658.

Work:

$$\begin{array}{r|l} -58.83 & 9.658 \\ \underline{57.95} & \overline{6.092} \\ -88 & \\ \underline{86} & \\ -2 & \\ \underline{2} & \end{array}$$

(1) Since 5883 is less than 9658, cross out the last digit of the divisor, 8, from the very start. The first digit of the quotient is 6. Multiply by 965, noting that the discarded digit gives 5 units ($6 \cdot 8 = 48$; drop 8 and round 4 to 5).

(2) Write the product 5795 under the dividend, digit place under digit place. The remainder is 88.

(3) Cross out the second from the last digit of the divisor, 5. The shortened divisor, 96, is not contained even once in the dividend 88; put a zero in the quotient, * do not multiply.

(4) There is no need to find the second remainder.

(5) Cross out one more digit of the divisor, 6. The shortened divisor, 9, is contained in the remainder 9 times, and so the third digit of the quotient is 9. Multiplying by the shortened divisor with account taken of the effect of the

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* Take careful note of this: a frequent mistake is made by not writing the zero and dropping the next digit of the divisor.

cross-lined digit, we have 86 and a remainder of 2. This is not the end of the operation. Dropping the last remaining digit, but taking into account its effect on the result, we find in the quotient another digit 2 ($2 \cdot 9 = 18$, the 8 is dropped and the 1 is rounded to 2). The last digit is obtained in simplest fashion by mentally bringing down zero to the last remainder 2; this yields $20:9 \approx 2$.

(6) A rough calculation gives the position of the decimal point. In the dividend and divisor we retain only the integral parts; it is clear that $58:9 \approx 6$, that is, the integral part of the quotient is a one-digit number. The result is therefore equal to 6.092, and not 60.92 or 6092, etc.

All digits of the answer are correct.

Example 2. 98.10:0.3216.

Work:

$$\begin{array}{r|l} -\,98.10 & 0.3216 \\ \underline{96.48} & \overline{305.0} \\ -\,1\,62 & \\ \underline{1\,61} & \\ 1 & \end{array}$$

(1) 9810 is greater than 3216. Multiply the first digit of the quotient, 3, by 3216 to get 9648.

(2) The remainder is 162.

(3) Cross-line the last digit, 6, of the divisor. The shortened divisor, 321, does not go even once into the remainder; the second digit of the result is zero.

(4) and (5) Cross-line another digit of the divisor, 1; the remainder 162 is divided by the shortened divisor 32; for the third digit of the quotient we have 5. Multiply it by 32 and take into account the effect of the discarded digit of the divisor to get 161. Subtract it from the remainder to get 1. Cross-line the digit 2 in the divisor. The shortened divisor 3 does not go into the remainder 1, and so the last digit of the quotient is zero.

(6) Set the decimal point on the basis of a rough rounding off of the given figures: taking 100 in place of 98.10 and 0.3 in place of 0.3216, we get $100:0.3 \approx 300$; hence the integral part of the quotient is a three-digit number. The quotient therefore is 305.0.

58. Involution and Evolution of Approximate Numbers

Raising a number to an integral power (involution) is simply iterated multiplication, and therefore everything stated in Secs. 54, 55 holds true. When raising a number to a small power, the result has as many correct digits as the original number or it contains a slight error in the last digit. If the degree is large, then the accumulation of small errors may affect higher digit places.

When extracting a root (evolution), the result has at least as many correct digits as the radicand. Thus, taking the square root of the approximate number 40.00, we can obtain four correct digits ($\sqrt{40.00} \approx 6.324$). *

The method for finding the square root of a number that is frequently taught in school is cumbersome and hard to remember, and its theoretical justification remains rather obscure to most students. Below we give a simple and easy-to-remember procedure for taking the square root of a number (to any required degree of accuracy). This method was described by the ancient Greek scholar Heron roughly two thousand years ago (Heron used common fractions, but we, naturally, will use decimals). The same method can be used to extract third and higher roots (see Sec. 58a below).

Rule for taking the square root. To find the square roots, make a reasonable guess as a first approximation and do as follows.

(1) Divide the radicand by the first approximation of the root; if it turns out that the quotient differs from the first approximation by a quantity that does not exceed the permissible error, the root is found.

(2) Otherwise we find the arithmetic mean (Sec. 59) of the divisor and quotient. This arithmetic mean yields a considerably more exact value (second approximation) of the root. If the choice of the first approximation has been felicitous, the second approximation yields three correct digits, ordinarily not less than four correct digits. Generally speaking, the number of correct digits is doubled in each new approximation.

(3) Subject the second approximation to the same test as the first: divide the radicand by the second approximation. If the accuracy of the result is not sufficient, then find the third approximation and proceed as before, etc.

Note 1. The foregoing method is "not afraid" of mistakes since it automatically corrects any error made in the preceding stage. The sole drawback is a slowing down of the computation process.

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* If we employ the procedure of extracting a square root usually studied in school, then we will have to annex four zeros to the radicand writing 40.000000, in order to obtain 6.324. The annexed zeros will be false digits, but the corresponding digits of the answer will be correct. The result will remain the same if in place of four zeros we annex four arbitrary numbers.

Example 1. $\sqrt{40.00}$. The radicand has four significant digits. There is no sence in finding more than four digits of the root, and so we will take four.

For the first approximation we need a number between 6 and 7 (since $6^2=36$ is less than the radicand, and $7^2=49$ exceeds it). Within these limits we can take any number, but if we want to save time and effort, we have to take a number less than 6.5 (since the radicand is much closer to 6^2 than to 7^2). Let us take 6.4 (we could have taken 6.3 or 6.2, but 6.1 is no good because 6.1 is too close to 6). Now do as follows.

(1) Divide the radicand 40.00 by the first approximation 6.4. We have $40.00:6.4=6.25$. It is already evident that the second digit of the quotient 6.25 differs from the dividend 6.4. This accuracy does not suffice.

(2) For the second approximation we take the arithmetic mean of the dividend 6.40 and the quotient 6.25 to get $(6.40+6.25):2=6.325$. We may expect that if not all four digits are correct in this second approximation, then at least the first three are.

(3) To check, divide the radicand 40.00 by the second approximation 6.325 (carry the division to the fourth digit): $40.00:6.325 \approx 6.324$. The quotient, 6.324, differs from the divisor, 6.325 by only one unit in the third decimal place, which means that the root has been found (to the required accuracy).

Indeed, squaring 6.324 (that is, multiplying that number by itself) we get a number less than the product of 6.324×6.325, which is 40.00 (approximately). Now if we square 6.325, we get a number greater than $6.325 \cdot 6.324 \approx 40.00$. Hence, the desired square root lies between 6.324 and 6.325, which means that it differs from 6.324 (or from 6.325) by less than one unit in the third decimal place: $\sqrt{40.00} \approx 6.324$ (all four digits are correct).

Example 2. $\sqrt{23.5}$. The desired root lies between 4 and 5 and is much closer to 5 than to 4 (since 23.5 is much closer to 25 than to 16). For the first approximation let us take the round number 5.0.

(1) Divide the radicand 23.5 by the first approximation 5.00 (carrying the quotient to the third digit): $23.5:5.0=4.70$.

(2) For the second approximation we take the arithmetic mean $(5.00+4.70):2=4.85$. We may expect that all three digits are correct.

(3) To check, divide the second approximation 4.85 into the radicand 23.5 to get $23.5:4.85 \approx 4.85$. Since the quotient is equal (to within the second decimal place) to the divisor, the root is found (to the highest possible degree of accuracy): $\sqrt{23.5} \approx 4.85$.

Note 2. If the radicand is a decimal fraction with one significant digit or zero in the integral part, then to find the first approximation it is advisable to move the decimal point to the right two, four, six, etc. digits so that the integral part has a small number of places. Then proceed as in Examples 1 and 2, and in the answer move the decimal point back one, two, three, etc. digits. The procedure is similar when the radicand has a multidigit whole-number portion; but then the decimal point is first moved to the left two, four, six, etc. digits.

In the radicand, the decimal point can only be moved an *even number of digits*.

Example 3. $\sqrt{0.008732}$. Move the decimal point 4 digits to the right: 87.32. In choosing the first approximation, we will take into account only the integral part. Let us take, say, 9.3.

(1) Divide 9.3 into 87.32. Carrying the division to the fourth significant digit, we get $87.32:9.3 \approx 9.389$.

(2) Find the arithmetic mean $(9.300+9.389):2 \approx 9.344$.

(3) To check, perform the division $87.32:9.344 \approx 9.345$. In either of the two numbers 9.344 and 9.345 all four digits are correct (the first number yields a deficit, the second, an excess).

(4) Since, at the beginning, we moved the decimal point rightwards 4 places, we now move it to the left (back) 2 places, and we have

$$\sqrt{0.008732} \approx 0.09344$$

Example 4. $\sqrt{8732000}$. Move the decimal point to the left 6 digits to get 8.732 (if we move it 4 digits, we get 873.2 and not 87.32 as in the previous example!). Take 3 as the first approximation.

(1) $8.732:3=2.911$.

(2) $(3.000+2.911):2=2.955$.

From the first operation it is clear that there were two correct digits in the first approximation (3.000). We therefore expect to have 4 correct digits in the second approximation. A check confirms this.

(3) Since we started by moving the decimal point 6 places to the left, we now come back three places: $\sqrt{8732000} \approx 2955$.

58a. Rule for Extracting a Cube Root

To extract the cube root of a number, make a plausible first estimate and proceed as follows.

(1) Divide by the first estimate (cf. rule of Sec. 58) twice: the dividend is first the radicand and then the number obtained by the first division. If the quotient (obtained by the second division) differs from the first estimate (approximation), that is to say, from the divisor, by a quantity not exceeding the permissible error, then the process is complete.

(2) Otherwise, average three numbers, namely the quotient (of two divisions) and the divisor taken twice (see Example 1 for an illustration of this second operation). We get a second approximation; which, if the first estimate was plausible enough, is correct to three digits, the fourth digit at worst requires a correction by unity.

(3) The second estimate can be tested in the same manner as the first, but this is a tiring procedure.

Example 1. $\sqrt[3]{785.0}$. The desired root lies between 9 and 10. Take 9.2 as a first estimate (since the radicand is roughly four times closer to 9^3 than to 10^3).

(1) Divide 9.2 into the radicand 785.0 and then into the quotient of 785.0:9.2. Instead, we can divide 785 by 9.2^2 to get 84.64. This yields

$$785.0:9.2:9.2 = 785.0:84.64 \approx 9.275$$

We see that the first estimate yields two correct digits. The best way to make the second estimate is to note that the radicand 785.0 is a product of three unequal factors: $785.0 = 9.2 \times 9.2 \times 9.275$, whereas we have to represent it as the product of three equal factors: $785.0 = x \cdot x \cdot x$ (where $x = \sqrt[3]{785.0}$). It is natural to assume that each of these equal factors should, approximately, be equal to the average of the factors 9.2, 9.2 and 9.275.

(2) Thus, for the second estimate we take the average $(9.275 + 9.200 + 9.200):3 = 9.225$. Compute by the short-cut method (see Sec. 60).

(3) For a check, divide the radicand 785.0 by the second estimate 9.225 and then divide the result again by 9.225 (or

divide the radicand by $9.225^2 \approx 85.09$). We get 9.225 (if an extra digit is not retained in the computations, we get 9.224):

$$\sqrt[3]{785.0} \approx 9.225 \text{ (correct to 4 figures)}$$

Note. In making the first estimate it is sometimes advantageous to move the decimal point in the radicand to the right (or to the left) 3, 6, 9, etc. places (cf. Sec. 58, Note 2). In the final result, move the decimal point back 1, 2, 3, etc. places. The decimal point may be moved only by as many digits as is divisible by 3.

Example 2. $\sqrt[3]{1835 \cdot 10}$. In the radicand, 18,350, move the decimal point three places to the left to get 18.35. This number is roughly midway between $2^3 = 8$ and $3^3 = 27$. So for the first approximation we take 2.5.

(1) Divide twice by 2.5 or, what is the same, once by 2.5^2. We get $18.35:2.5:2.5 = 18.35:6.25 \approx 2.94$.

We see that in this first approximation only one digit is correct. Thus, we must expect that in the second approximation there will only be two correct digits. Therefore, in the next operation we carry the answer only to two places.

(2) For the second estimate take the average $(2.5+2.5+2.9):3 \approx 2.6$.

(3) To make the result more precise we divide twice by 2.6 to get

$$18.35:2.6:2.6 = 18.35:6.76 \approx 2.715$$

We see that the second estimate furnished two correct digits, and so the third most likely will yield 4 correct digits.

(4) For the third estimate, average

$$(2.715+2.600+2.600):3 = 2.638$$

A check (which we omit) would show that the result is correct to four significant digits:

$$\sqrt[3]{1835 \cdot 10} \approx 26.38$$

59. Mean Quantities

If we have a sequence of quantities (numbers), any one between the smallest and greatest is a mean. The most frequently used mean quantities are the arithmetic mean and the geometric mean.

The *arithmetic mean* (arithmetic average, or, simply, average) is obtained by adding the given quantities and dividing the sum by the number of quantities:

$$\text{a. m.}=\frac{a_1+a_2+\ldots+a_n}{n}$$

(a_1, a_2, ..., a_n are the given quantities, n is the total number of quantities).

Example. Given the numbers 83, 87, 81, 90.

$$\text{a. m.}=\frac{83+87+81+90}{4}=85\frac{1}{4}$$

The *geometric mean* is obtained by multiplying the given quantities and taking the nth root of the product (where n is the index of the root and is equal to the number of quantities taken):

$$\text{g. m.}=\sqrt[n]{a_1a_2\ \ldots\ a_n}$$

(a_1, a_2, ..., a_n are the given quantities, n is the total number of them).

Example. Given the numbers 40, 50, 82.

$$\text{g. m.}=\sqrt[3]{40\cdot 50\cdot 82}=\sqrt[3]{164{,}000}\approx 54.74$$

The geometric mean is always less than the arithmetic mean (average) except for the case when all numbers are equal. Then the arithmetic mean is equal to the geometric mean. When there are only small fractional differences between the numbers, the difference between the arithmetic mean and the geometric mean is small compared to the numbers.

Averaging (computing arithmetic means) is of great importance in all branches of practical work.

Example 1. The distance between two points is measured with a 10-metre tape measure having centimetre divisions. Ten measurements are taken, which, in metres, are: 62.36, 62.30, 62.32, 62.31, 62.36, 62.35, 62.33, 62.32, 62.38, 62.37. The diversity of results is due to accidental inaccuracies in the measuring process. These findings are then averaged:

$$\begin{aligned}\text{a. m.}=&(62.36+62.30+62.32.+62.31+62.36+62.35\\&+62.33+62.32+62.38+62.37):10=62.34\end{aligned}$$

This number is a more reliable value of the distance than the numbers obtained in the measurements because random (accidental) errors are nearly always balanced when computing the average (see Sec. 61 below).

Example 2. Height measurements are taken of 1000 persons. These are averaged. The result is the "average height" of the persons involved. It does not, generally speaking, signify the actual height of a given person. But if the measurements are taken of a large number of other people, the average height will again be just about the same. Quite naturally, it may happen that either giants or pigmies predominate in a sample of 1000. However, out of all conceivable cases these exceptional ones constitute an insignificantly small percent. Hence, for all practical purposes we can take it that the average height of any group of 1000 persons will be almost the same. The arithmetic means obtained in mass measurements are termed *statistical means*. Statistical means are of considerable practical importance. For example, knowing the average milk yield of a cow of a definite breed under specific feeding conditions, etc., it is possible to compute the yield of a herd by multiplying the average yield by the number of cows in the herd.

60. Abridged Calculation of the Arithmetic Mean

The numbers involved in computing an arithmetic mean are ordinarily bunched rather close together. If such is the case, computing the average (arithmetic mean) can be greatly simplified by the following technique.

(1) Take an arbitrary number close to the given numbers. If the given numbers differ from one another in the last digit alone, it is preferable to make the last digit of the selected number zero; if the given numbers differ in the last two digits, it is convenient to take a number ending in two zeros, and so on.

(2) Subtract this number in succession from each of the given ones.*

(3) Take the arithmetic mean of the differences thus found.

(4) Add the mean to the chosen number.

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* Both positive and negative numbers can be expected (see Sec. 67 on negative numbers). To avoid this, take a number less than any of the given ones. However, the computations will be somewhat simpler if the chosen number is about midway between the extremes of the given numbers.

Example. Find the arithmetic mean of ten numbers: 62.36, 62.30, 62.32, 62.31, 62.36, 62.35, 62.33, 62.32, 62.38, 62.37 (cf. with the preceding example).

(1) Choose the number 62.30.

(2) Subtract 62.30 from the given numbers; we find the differences (in hundredths) to be 6, 0, 2, 1, 6, 5, 3, 2, 8, 7.

(3) Now take the average of these differences, which is 4 (hundredths).

(4) Add 0.04 to 62.30 to get 62.34. This is the desired arithmetic mean (average).

61. Accuracy of the Arithmetic Mean

If the arithmetic mean is obtained from a comparatively small sequence of measurements (say, 10, like in Example 1, Sec. 59), it might well be that the actual value is somewhat different from the computed average. Then it is important to know how great this deviation can be. We are not speaking of the theoretically conceivable deviation (which can be arbitrarily great) but of the practically possible deviation (cf. Example 2, Sec. 59). The magnitude of the latter depends on the magnitude of the so-called *root mean square deviation.*

The root mean square deviation is the square root of the arithmetic mean of the squares of the deviations from the mean. It is denoted generally by the Greek letter σ (sigma):

$$\sigma=\sqrt{\frac{(a_1-a)^2+(a_2-a)^2+\ldots+(a_n-a)^2}{n}} \qquad \text{(A)}$$

where $a=(a_1+a_2+\ldots+a_n):n$ (here $a_1, a_2, \ldots, a_n$ are the given numbers, n is the total number of them, a is their arithmetic mean, and σ is the root mean square deviation).

Note. In formula (A) any one of the differences may be replaced by its reciprocal; this enables one to dispense with negative numbers (see Sec. 67 on negative numbers). Namely, when one of the given numbers is less than the mean, we take it for the subtrahend, and the mean for the minuend.

Example. Compute the root mean square deviation for the numbers of the preceding section. There, we found the mean to be 62.34. The deviations from the numbers 62.36, 62.30, etc. and their arithmetic mean are (in hundredths): 2, 4, 2, 3, 2, 1, 1, 2, 4, 3. The squares of these deviations are 4, 16, 4, 9, 4, 1, 1, 4, 16, 9. The arithmetic mean of the squares of the deviations is

$$\frac{4+16+4+9+4+1+1+4+16+9}{10}=6.8$$

(hundredths). The square root of this number, $\sqrt{6.8} \approx 3$ (hundredths); $\sigma = 0.03$.

If the number of measurements is approximately equal to 10, then the true value of the quantity cannot deviate from the arithmetic mean by more than the root mean square deviation σ. To be more precise, deviations exceeding σ are possible only in exceptional cases, the number of which comes to about half a percent of all possible cases. In the example just considered, the true value cannot, practically speaking, deviate from the number 62.34 by more than 0.03. It therefore lies between $62.34 - 0.03 = 62.31$ and $62.34 + 0.03 = 62.37$.

If the number of measurements is substantially greater than ten, then the maximum practically possible deviation of the true value from the arithmetic mean will be less than σ. Namely, the deviation will not exceed the value $\frac{3\sigma}{\sqrt{n}}$ (where n is the number of measurements). Thus, when the number of measurements is roughly 1000, the only practically possible deviations are those that do not exceed 0.1σ.

62. Ratio and Proportion

The quotient of two numbers is termed their *ratio*. The term "ratio" was once applied only to cases when it was required to express one quantity as a fraction of another (homogeneous with the first); say, one length as a fractional part of another, one area as a fraction of another area, and so forth. These problems are handled by division (see Sec. 38). This explains why the special term "ratio" appeared: it once had a different meaning from "division", which referred to the division of a denominate number by an abstract number. This distinction is no longer made; for instance, we speak of the ratio of nonhomogeneous quantities, say the weight of a solid to its volume, etc. When speaking of homogeneous quantities, we often use percentage.

Example. A library has 10,000 books, of which 8000 are in Russian. What is the ratio of Russian books to the total number? $8000 : 10{,}000 = 0.8$. The desired ratio is 0.8, or 80%.

The dividend is called the *antecedent* of the ratio, the divisor, the *consequent*. In our example, 8000 is the antecedent, and 10,000 the consequent.

Two equal ratios form a *proportion*. Thus, if one library has 10,000 books, of which 8000 are Russian and another library has 12,000 books, of which 9600 are Russian, then the ratio of Russian books to the total number of books in both libraries is the same: $8000:10{,}000 = 0.8$; $9600:12{,}000 = 0.8$. What we have is a proportion, which we write as follows: $8000:10{,}000 = 9600:12{,}000$. In words, we say that 8000 is to 10,000 as 9600 is to 12,000. 8000 and 12,000 are the *extremes of the proportion*, and 10,000 and 9600 are the *means, of the proportion*.

The product of the means equals the product of the extremes. In our example, $8000 \cdot 12{,}000 = 96{,}000{,}000$; $10{,}000 \cdot 9600 = 96{,}000{,}000$. One of the extremes is equal to the product of the means divided by the other extreme. In the same way, one of the means is equal to the product of the extremes divided by the other mean. If

$$a:b = c:d$$

then

$$a = \frac{bc}{d}, \quad b = \frac{ad}{c}$$

and so on. In our example

$$8000 = \frac{10{,}000 \cdot 9600}{12{,}000}$$

This property is always used to compute the missing term of a proportion when the three other terms are known.

Example. $12:x = 6:5$ (x is the missing term, the unknown). $x = \frac{12 \cdot 5}{6} = 10$.

For practical applications of proportions see Sec. 64.

A proportion in which the means are equal is termed a *continued proportion*; for example, $18:6 = 6:2$. The mean term of a continued proportion is the *geometric mean* (see Sec. 59) *of the extreme terms*. In our example, $6 = \sqrt{18 \cdot 2}$.

63. Proportionality (Variation)

The values of two different quantities can be interdependent. Thus, the area of a square depends on the length of the side, and conversely, the length of the side of a square is dependent on the area of the square.

Two mutually dependent quantities are termed *proportional* if the ratio of their values remains constant.

Example. The weight of kerosene is proportional to its volume; 2 litres of kerosene weigh 1.6 kg, 5 litres weigh

4 kg, 7 litres weigh 5.6 kg. The ratio of the weight to the volume is $\frac{1.6}{2}=0.8$, $\frac{4}{5}=0.8$, $\frac{5.6}{7}=0.8$, etc.

The constant ratio of proportional quantities is called the *constant of proportionality (or constant of variation, or proportionality factor)*. It shows how many units of one quantity there are for every unit of another quantity; in our example, the number of kilograms that 1 litre of kerosene weighs (the specific weight of kerosene).

If two quantities are proportional, then any pair of values of one quantity forms a proportion with a pair of the corresponding values of the other taken in the same order. In our example, $1.6:4=2:5$, $1.6:5.6=2:7$, etc. Accordingly, we can define proportionality as follows: two quantities that depend on each other so that any increase in one causes an increase (in the same ratio) in the other are called *proportional quantities.*

If the dependence of two quantities is such that one increases as the other decreases (in the same ratio) then we have *inversely proportional quantities*. For example, the running time of a train between two stations is inversely proportional to the speed of the train. At a speed of 50 km/h, a train covers the distance between Moscow and Leningrad in 13 hours; going at 65 km/h, it covers the distance in 10 hours; that is when the speed increases in the ratio $\frac{65}{50}=\frac{13}{10}$, the running time diminishes in the ratio $\frac{13}{10}$.

If two quantities are inversely proportional, then any pair of values of one quantity forms a proportion with a pair of the corresponding values of the other taken in the reverse order. In our example, $65:50=13:10$.

The product of the values of two inversely proportional quantities remains unchanged. In our example, $50\cdot 13=650$; $65\cdot 10=650$ (650 km is the distance between Moscow and Leningrad).

64. Uses of Proportions. Interpolation

The solution of many problems involves proportional quantities. Application of the rules given in Sec. 62 mechanizes the solution of such problems, reducing them to a unified procedure, as illustrated in the examples given below.

Example 1. Fuel consumption at a factory was at 1.8 tons per 24 hours and at a cost of 3000 roubles per year.

Efficiency proposals reduced daily consumption to 1.5 tons. What should the budget plan of the next year be with respect to fuel consumption at this factory?

An unsophisticated solution to this problem would be:

(1) find the annual fuel consumption prior to the efficiency proposals: $1.8 \cdot 365 = 657$ (tons);

(2) find the cost of one ton of fuel: $3000 : 657 = 4.57$ (roubles);

(3) find the annual cost of the fuel following the introduction of the efficiency suggestion:

$$4.57 \cdot 1.5 \cdot 365 = 2500 \text{ (roubles)}$$

A much faster and easier solution can be found by noting that the daily consumption of fuel and the annual cost are proportional quantities (this is evident from the fact that an increase in the daily consumption produces a proportionate increase in the yearly cost; see Sec. 63).

Scheme of solution:

1.8 tons 3000 roubles,
1.5 tons x roubles,

$$x : 3000 = 1.5 : 1.8,$$

$$x = \frac{3000 \cdot 1.5}{1.8} = 2500 \text{ roubles}$$

Although proportional relationships are encountered very often, most of the relationships in practical situations do not obey the law of proportionality. It is therefore all the more important to note that even for such quantities the procedure of computation using proportions is still meaningful. For instance, if we consider the variations of nonproportional quantities within a certain rather narrow range, these changes will, for all practical purposes, be proportional.

To illustrate, take a square. A side is not proportional to the area; for instance, a side of 2 m is associated with an area of 4 m^2; a side of 2.01 m, with an area of $(2.01)^2 = 4.0401 \approx 4.040$ (m^2); a side of 2.02 m, with an area of $4.0804 \approx 4.080$ (m^2), and so on. We thus see that the ratio of the sides (for instance, $2.01 : 1$) is not equal to the ratio of the corresponding areas ($4.040 : 1$). However, the ratio of the *changes in the side* within the range we confine ourselves to is, for all practical purposes, equal to the ratio of the *changes in the area*.

Indeed, when the side increases from 2 m to 2.01 m, the change is 0.01 m, when it increases from 2 to 2.02 m, the change is 0.02 m. The ratio of the changes 0.02:0.01 is 2. The corresponding changes in the area will (to within the third decimal) be: in the first case, 0.040, in the second 0.080. The ratio of the changes, 0.080:0.040, is also 2. Thus, the change in length is proportional to the change in area if the quantities are taken to three places of decimals. If a fourth decimal is taken, a slight deviation from proportionality will be noted. To avoid any deviation even in the fourth decimal place, regard the change in a side over a still smaller range (say from 1 m to 1.002 m instead of from 1 to 1.02 m). In practical work, we always have regard only for a definite number of decimals (three, four, and rarely five). For this reason we can consider the changes in the side and area of a square as proportional quantities. The situation is the same in an overwhelming majority of other cases. This circumstance is utilized in reading between the lines, so to speak, when we have to do with tables covering a relatively small number of data. We are able to pick out values which lie between the tabulated values.

Example 2. Take a table of square roots (see pages 14-17). Suppose we wish to find $\sqrt{63.2}$. The table does not have the number 63.2, but only 63, 64, 65, etc.

Radicand	Square Root	Change in Square Root
63	7.937	
64	8.000	0.063
65	8.062	0.062

We calculate (see third column) the change, in the value of the root when the radicand varies by unity from 63 to 64 and from 64 to 65. We see that the difference in these changes occurs only in the third decimal place (by one unit). Actually this difference is still less; it occurs only in the fourth decimal place and rounding off to three decimals gave rise to it in the third decimal place.

Now if we take only three decimals, all our changes will be just about the same, that is to say, within the range between 63 and 65 the changes in square roots taken to three decimals are proportional to the changes in the radicands. We therefore find $\sqrt{63.2}$ using the following scheme:

Change in Radicand	Change in Square Root
1	0.062
0.2	x

$$x:0.062=0.2:1,$$

$$x=\frac{0.062\cdot 0.2}{1}=0.012$$

Now we find $\sqrt{63.2}$ by adding to $\sqrt{63}\approx 7.937$ the number 0.012. This yields

$$\sqrt{63.2}\approx 7.949$$

Check by extracting the root to three decimals and you will see that all the decimals of our result are correct.

The procedure we have just discussed is called *interpolation* (an inserting between). In mathematics, interpolation signifies any procedure by means of which it is possible, in a table with a given number of tabulated values, to find certain intermediate numerical values not directly given in the table. The elementary kind of interpolation which we discussed above is termed *linear interpolation*.

Interpolation is extensively used when dealing with tables of almost any kind.

ALGEBRA

65. The Subject of Algebra

The subject of algebra involves the study of equations (Secs. 79-81) and a number of other problems that developed out of the theory of equations. At the present time, when mathematics has split up into a number of specialized areas, the field of algebra includes only equations of a special kind, the so-called *algebraic equations* (see Sec. 83).* On the origin of the name "algebra" see Sec. 66.

66. Historical Survey of the Development of Algebra

Babylonia. The roots of algebra go deep into antiquity. About 4000 years ago, Babylonian scholars were already solving quadratic equations (Sec. 93) and systems of two equations, one of which was of second degree (Sec. 97). These equations were used in solving a diversity of problems in land measurement, construction of buildings and in military affairs.

The literal designations which we use today in algebra were unknown to the Babylonians who formulated their equations rhetorically.

Greece. The first syncopated (abridged) notations for unknown quantities are encountered in the writings of the ancient Greek mathematician Diophantus (2nd to 3rd century). For the unknown, Diophantus used the word "arithmos" (number), the second power of the unknown was denoted by "dunamis" (the word had many meanings: power, property, degree **). For the third power, Diophantus used the term

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* Note that the usual school course of algebra includes areas that are only remotely related to the theory of equations. Such, for example, as progressions and logarithms, which belong more to arithmetic than to algebra. Their inclusion in the course of algebra is justified on pedagogical grounds.

** "Dunamis" was translated into Arabic as "mal" meaning property. In the 12th century mathematicians in Western Europe translated "mal" into Latin as "census", which has the same meaning.

"kubos" (cube), for the fourth power we find (translated into English) square-square, for the fifth, square-cube, for the sixth, cube-cube. He denoted these quantities by the first letters of the corresponding names (we give them in Latin letters): *ar*, *du*, *cu*, *ddu*, *dcu*, *ccu*. To distinguish the unknowns from known quantities, the latter were accompanied by the designation "mo" (monades for "units"). Addition was not indicated in any way, an abbreviation was used for subtraction, equality was shown by "is" ("isos" means equal).

Neither the Babylonians nor the Greeks considered negative numbers. An equation like 3 *ar* 6 *mo is* 2 *ar* 1 *mo* ($3x+6=2x+1$) Diophantus called "inappropriate". When Diophantus transposed terms from one side of the equation to the other, he said that an addend becomes a subtrahend, and a subtrahend becomes an addend.

China. Chinese scholars were solving first-degree equations and systems of them and also quadratic equations 2000 years prior to the Christian era. They were acquainted with negative numbers and irrational numbers. Since each symbol in Chinese writing stands for a concept, there could be no syncopations in Chinese algebra.

At later periods, Chinese mathematics was enriched with new attainments. At the end of the 13th century, the Chinese were fully acquainted with the law of formation of binomial coefficients which today goes by the name of "Pascal's triangle" (see Sec. 136). In Western Europe this law was discovered by Stifel, 250 years later.

India. Hindu scholars made extensive use of syncopated notation for unknown quantities and their powers. These notations were the initial letters of the corresponding names (an unknown was called "so-much"; the names of various colours—black, blue, yellow, etc.—were used for a second, third, etc. unknown). Hindu scholars made much use of irrational and negative numbers (Greek mathematicians knew how to find approximate values of roots but eschewed irrationalities in algebra). A new addition to the family of numbers was zero, which came in with the negative numbers. Formerly it had been used solely for the absence of a number, as a placeholder.

Arab-language countries. Uzbekistan, Tajikistan. The Hindu authors wrote on algebraic problems in their astronomical works. It was in Arabic writings—the international language of the Muslim world—that algebra emerged as an independent discipline. The founder of algebra as a special branch of learning was the Central Asian scholar Mohammad of Khorezmi, more generally known as al-Khowarizmi (dweller of Khorezmi). His algebraic work, composed in the 9th century A. D., bears the name "the science of the reunion and opposition", or, more freely, "the science of transposition and cancellation". "Transposition" denoted the transfer of a subtrahend from one side of the equation to the other where it becomes an addend; the "opposition" (or cancellation) was the gathering of unknowns on one side of the equation and the unknowns on the other side. The Arabic for "transposition" is "al-jabr". Whence the name "algebra".

Al-Khowarizmi and those that followed him made extensive use of algebra in commercial and monetary computations. Neither he nor any of the other mathematicians who wrote in Arabic made any use of abbreviations.* Neither did they recognize negative numbers. From Hindu sources they knew about negative numbers but considered their use insufficiently justified. This was true, but whereas the Hindu scholars were able to confine themselves to a single complete quadratic equation, al-Khowarizmi and his successors had to distinguish three cases ($x^2+px=q$, $x^2+q=px$, $x^2=px+q$, where p and q are positive numbers).

Central Asian, Persian and Arabic mathematicians enriched algebra in a variety of ways. In higher-degree equations they knew how to find approximate values of the roots to a very high degree of accuracy. Thus, the celebrated Central Asian philosopher, astronomer and mathematician al-Biruni (973—1048), also of Khorezmi, reduced the problem of computing the side of a regular nonagon inscribed in a given circle to the cubic equation $x^3=1+3x$ and found (in sexagesimal fractions) the approximate value $x = 1.52'45''47'''13''''$ to within $\frac{1}{60^4}$, which in decimal fractions is correct to seven decimal places. (The sexagesimal fraction can be read as one unit, 52 sixtieths, 45 three thousand six hundredths, etc.). The scholar Omar Khayyam (1036—1123) of Naishapur, the famous classic of Iranian and Tajik poetry (known in the West for his celebrated Rubaiyat) made a systematic study of equations of the third degree. Neither he nor any other of the mathematicians of the Muslim world were able to find expressions for the roots of a cubic equation in terms of the coefficients. However, Omar

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* Abbreviations were hardly necessary since Arabic writing is brief: vowels are not written, consonants and semi-consonants are simple letters and often merge into a single symbol.

Khayyam developed a method by which it is possible (geometrically) to find the number of real roots of a cubic equation (he himself was only interested in positive roots).

Medieval Europe. In the 12th century, the "Algebra" of al-Khowarizmi was translated into Latin and studied in Europe. It marked the beginning of the development of algebra in European countries, at first under the strong influence of the science of the East. Syncopated notation appeared for unknowns and new problems involved in trading were solved, but no essential advances were made until the first third of the 16th century when the Italians del Ferro and Tartaglia found rules for solving cubic equations of the type $x^3 = px + q$, $x^3 + px = q$, $x^3 + q = px$, and Cardano, in 1545, demonstrated that any cubic equation can be reduced to one of these three types. At the same time, Ferrari, a pupil of Cardano, solved a quartic (fourth-degree) equation.

The rules for solving such equations were so complicated that improvements had to be made in notation. These took place gradually during a whole century. At the end of the 16th century, the French mathematician Viète introduced literal symbols for unknowns and for known quantities as well (the unknowns were denoted by capital vowels, the known quantities by capital consonants). Syncopated notation was introduced for operations as well. Different authors used different kinds. By the middle of the 17th century, algebraic symbolism, thanks to the efforts of the French scholar Descartes (1596—1650), took on the general outlines that it has today.

Negative numbers. During the 13th to 16th centuries, negative numbers were considered by Europeans only in exceptional cases. After the discovery of the solution of the cubic equation, negative numbers gradually came to be accepted in algebra, although they were called "false" numbers. In 1629 Girard (France) gave a geometric depiction of negative numbers that we still use today. About twenty years later, negative numbers were accepted generally.

Complex numbers. The introduction of complex numbers (Secs. 92, 98) was also connected with the discovery of the solution to the cubic equation.

Even before this discovery, in the solution of the quadratic equation $x^2 + q = px$ one encountered a case when it was required to find the square root of $\left(\frac{p}{2}\right)^2 - q$, where the quantity $\left(\frac{p}{2}\right)^2$ was less than q. It was concluded in that case that the equation does not have any solutions. Of course, it is hard to imagine that the new (complex) numbers would be introduced at a time when even negative numbers were considered "false". Yet when solving a cubic equation by the Tartaglia rule it turned out that without operations involving imaginary numbers it is impossible to obtain a real root.

Let us go into this in more detail. By Tartaglia's rule a root of the equation

$$x^3 = px + q \tag{1}$$

is given by the expression

$$x = \sqrt[3]{u} + \sqrt[3]{v} \tag{2}$$

where u and v are solutions of the system

$$u + v = q, \quad uv = \left(\frac{p}{3}\right)^3 \tag{3}$$

For example, for the equation $x^3 = 9x + 28$ ($p = 9$, $q = 28$) we have

$$u + v = 28, \ uv = 27$$

whence either $u = 27$, $v = 1$ or $u = 1$, $v = 27$, In both cases

$$x = \sqrt[3]{27} + \sqrt[3]{1} = 4$$

This equation does not have any other real roots.

But, as Cardano had noted, the system (3) may not have any real solutions, whereas equation (1) has a real root and, what is more, a positive root. Thus, the equation $x^3 = 15x + 4$ has the root $x = 4$, but the system

$$u + v = 4, \ uv = 125$$

has the complex roots: $u = 2 + 11i$, $v = 2 - 11i$ (or $u = 2 - 11i$, $v = 2 + 11i$).

Bombelli (1572) was the first to shed light on this mysterious phenomenon. He pointed out that $2 + 11i$ is the cube of $2 + i$, and $2 - 11i$ is the cube of $2 - i$; hence we can write $\sqrt[3]{2 + 11i} = 2 + i$, $\sqrt[3]{2 - 11i} = 2 - i$ and then formula (2) yields $x = (2 + i) + (2 - i) = 4$.

It was now impossible to ignore complex numbers. However, the theory of complex numbers developed slowly. As late as the 18th century, famous mathematicians argued about how to find the logarithms of complex numbers.

Although complex numbers helped to obtain a wide range of important facts involving real numbers, their very existence seemed doubtful. Exhaustive rules for operating with complex numbers were given in the middle of the 18th century by Euler, one of the greatest mathematicians in the history of science. At the turn of the 19th century, Wessel of Denmark and Argand of France gave a geometrical representation of complex numbers (Sec. 104; the first steps in this direction were taken by Wallis of England in 1685). But the work of Wessel and Argand was disregarded and only in 1831, when this method was developed by the great German mathematician Gauss, was it accepted generally.

After solutions had been found for equations of the third and fourth degree, mathematicians strenuously sought the formula for solving the quintic (fifth-degree) equation. However, Ruffini (Italy) proved, at the turn of the 19th century, that the literal fifth-degree equation $x^5+ax^4+bx^3+cx^2+dx+e=0$ cannot be solved algebraically; more precisely, it is impossible to express any root of it in terms of the literal quantities a, b, c, d, e using the six algebraic operations of addition, subtraction, multiplication, division, involution and evolution (Ruffini's proof was not without fault, and in 1824 Abel of Norway gave a flawless proof).

In 1830 Galois (France) demonstrated that no general equation whose degree exceeds 4 can be solved algebraically.

Nevertheless, every nth-degree equation has (if we consider complex numbers as well) n roots, some of which may be equal. This was known to mathematicians as early as the 17th century (it stemmed from the analysis of numerous particular cases), but only at the end of the 18th century was the theorem mentioned above proved by Gauss.

The problems that engaged algebraists in the 19th and 20th centuries for the most part go beyond the range of elementary mathematics. Suffice it to note that in the 19th century many methods were developed for approximate solution of equations. In this direction, important results were obtained by the great Russian mathematician N. I. Lobachevsky.

67. Negative Numbers

The first numbers known to man were the natural numbers (Sec. 16). But these numbers do not suffice even in the simplest cases. Indeed, in the general case, one natural number cannot be divided by another if we confine ourselves to natural numbers alone. Yet situations arise in which we have to divide, say, 3 by 4, 5 by 12, and the like. Without the introduction of fractions, the division of natural numbers is impossible; fractions make this operation possible.

But the operation of subtraction still remains impossible in certain cases even after the introduction of fractions: we cannot subtract a larger number from a smaller number, say 5 from 3. In everyday life we do not need to perform such subtraction and so for a very long time that operation was considered not only impossible but even senseless.

The development of algebra demonstrated that such an operation is necessary (see Sec. 68 below) and it was put to use by scholars of India in about the 7th century, by Chinese scholars earlier still. Hindu scholars, seeking to find instances of such subtraction, came to an interpretation of it from the point of view of trade transactions. If a merchant has 5000 roubles and buys 3000 roubles worth of goods, he has $5000-3000=2000$ roubles. But if he has 3000 and buys 5000 roubles worth of goods, then he is 2000 roubles in debt. It was considered, accordingly that we subtract 5000 from 3000, the result being a dotted 2000, which meant "two thousand in debt".

This interpretation was artificial because a merchant never found the sum of his debt by subtracting $3000-5000$, he always performed the subtraction thus: $5000-3000$. What is more, this could serve—with a stretch of the imagination—only to explain the rules of addition and subtraction of "dotted numbers", but it could not account for the rules of multiplication and division (see Sec. 69 on rules of operations). Still and all, this interpretation remained for a long time in textbooks and manuals, and even today it occasionally appears.

The "impossibility" of subtracting a number from a smaller one is due to the fact that the natural numbers (positive whole numbers) form an infinite sequence in only one direction. If we subtract successively 1 from, say, the number 7, we get

$$6,\ 5,\ 4,\ 3,\ 2,\ 1$$

One more subtraction yields an absence of any number, and from there on there is nothing to subtract from. If we want to make subtraction possible in all cases, we must: (1) consider the "absence of a number" as a number (zero); (2) consider it possible to subtract another unit from this number, etc.

We thus generate new numbers which today look like this:

$$-1, -2, -3, \text{etc.}$$

These numbers are termed *negative integers (or negative whole numbers)*. The minus sign reminds us of the origin of a negative number from the successive subtraction of unity. This sign is called a sign of quantity to distinguish it from the subtraction sign, which is called an operational sign.

The introduction of negative integers implies the need for negative fractional numbers. If we take it that $0-5=-5$, then we must also accept the fact that $0-\frac{12}{7}=-\frac{12}{7}$. The number $-\frac{12}{7}$ is a negative fractional number.

In contradistinction to negative numbers (integral and fractional), the numbers (integral and fractional) which are studied in arithmetic are called *positive numbers*. To bring out this distinction still more, we can affix a plus sign to any positive number, in which case this is a sign of quantity (and not the sign of an operation). For example, the number 2 can be written as $+2$. *

Together, negative and positive numbers are called signed numbers in school textbooks. The generic term in scientific terminology for these numbers, together with zero, is *rational numbers*. The meaning of this term will become clear when we discuss the concept of an irrational number (see Sec. 91). Just as, prior to the introduction of negative numbers, there were no positive numbers and the number $\frac{3}{4}$ was simply a fraction and not a positive fraction, so prior to the introduction of irrational numbers, the numbers $+5$, -5, $-\frac{3}{4}$, $+\frac{3}{4}$, etc., were simply positive and negative numbers and fractions, and not rational numbers.

68. Negative Numbers (History and Rules of Operation)

For the student, probably the most difficult item in algebra is that devoted to operations involving negative numbers. This is not because the rules are complicated.

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* The fact that the signs of operations and the signs of quantity are the same (+ and −) is an advantage with respect to computations, but the beginner finds this rather complicated. It is therefore advisable, at the beginning, to distinguish between the operational sign and the sign of quantity, and write $\overline{2}$ for a negative two instead of -2. This is done in logarithmic computations (see Sec. 129 et seq.).

Quite the contrary, they are simple. The difficulty is twofold: (1) Why introduce negative numbers? (2) Why are the operations involving them what they are and not something else? For instance, it is often hard to grasp why multiplication and division of a negative number by a negative number yields a positive number.

These questions usually arise because negative numbers are usually introduced before equations, and the rules for operating with negative numbers are not re-examined. Actually, it is in connection with the solution of equations that both questions can be answered. Historically speaking, that is exactly how the negative numbers arose. If there had been no equations, there would not have been any need for negative numbers.

For a long time equations were studied without the aid of negative numbers. This was extremely inconvenient. It was to overcome these inconveniences that negative numbers were introduced. It is worth noting that for a long time many outstanding mathematicians refused to use them and only grudgingly gave way to the inevitable. Even Descartes was still calling them "false numbers".

A simple example will serve to illustrate the nature of these inconveniences. When solving an equation of the first degree in one unknown, say,

$$7x-5=10x-11$$

we transpose the terms so that the knowns are on one side and the unknowns are on the other. In such an operation, the signs are reversed. Collecting unknown quantities in the right member of the equation and known quantities in the left member, we get

$$11-5=10x-7x, \quad 6=3x, \quad x=2$$

These manipulations can be carried out without invoking any negative numbers at all and considering the + and − signs as signs of addition and subtraction, and not as the signs of positive and negative numbers. But then we have to think over the question of which side to move the unknown terms, because if, say, in the above equation the unknown terms are transposed to the left side, we get

$$7x-10x=5-11$$

Without negative numbers, we cannot subtract 11 from 5, neither can we subtract $10x$ from $7x$, which means we cannot get ahead with the solution. Now it is not always so easy to see, beforehand, how to avoid this situation, especially if there are a large number of terms. A computor has to be ready to do a double amount of work in transposing terms to the proper side. It was to make the computational process more efficient that negative numbers were introduced. Indeed, if we agree to consider "possible" the "impossible" subtraction $5-11$, and denote the result by -6, and make the subtraction $7x-10x$ yield $-3x$, then we obtain

$$-3x=-6$$

Whence $x=-6:-3$.

Now it turns out that when introducing negative numbers, we have to set up the rule that in the division of a negative number (-6) by a negative number (-3), the quotient is a positive number (2). This is so because the quotient must yield the value of the unknown quantity x, which was found earlier in a different manner (without using negative numbers) and proved to be equal to 2.

That, in rough outline, was how negative numbers were first introduced: the aim was to rationalize the process of computation. The rules involving negative numbers emerged from the introduction of this more efficient technique into computational procedures.

Numberless tests and years of using negative numbers have demonstrated the extreme effectiveness of this technique which has found brilliant applications in all spheres of science and engineering. Everywhere, the introduction of negative numbers permits embracing, in a single rule, phenomena that would require dozens of rules if we confined ourselves to positive numbers.

To summarize, the two questions posed above may be answered as follows: (1) negative numbers were introduced so as to dispose of certain difficulties arising principally in the solution of equations; (2) the rules involving them follow from the necessity to coordinate the results obtained by means of negative numbers with those obtained without them.

All these rules (see Sec. 69) can be established when considering the most elementary kinds of equations, in the same way that we set up a rule for the division of negative numbers.

69. Operations with Negative and Positive Numbers

The absolute value of a negative number is the positive number obtained simply by reversing the sign. The absolute value of -5 is $+5$, that is, 5. The absolute value of a positive number (zero included) is the number itself.

The absolute-value sign consists of two vertical bars that enclose the number whose absolute value is being taken. For example, $|-5|=5$, $|+5|=5$, $|0|=0$.

1. Addition. (a) To add two numbers with like signs, combine their absolute values and prefix the common sign.

Examples. $(+8)+(+11)=19$, $(-7)+(-3)=-10$.

(b) To add two numbers with unlike signs, find the difference between their absolute values and prefix the sign of the number whose absolute value is greater.

Examples. $(-3)+(+12)=9$, $(-3)+(+1)=-2$.

2. Subtraction. The subtraction of one number from another can be replaced by addition; in this case, the minuend retains its sign, and the sign of the subtrahend is reversed.

Examples.

$$(+7)-(+4)=(+7)+(-4)=3,$$
$$(+7)-(-4)=(+7)+(+4)=11,$$
$$(-7)-(-4)=(-7)+(+4)=-3,$$
$$(-4)-(-4)=(-4)+(+4)=0$$

Note. When performing addition and subtraction, especially when the operation involves several numbers, it is advisable to do as follows: (1) remove all brackets; to do this, affix a plus sign if the earlier sign in front of the brackets was the same as that inside the brackets, and a minus sign if it was opposite to that inside the brackets; (2) add the absolute values of all the numbers which now have the plus sign; (3) add the absolute values of all the numbers which now have the minus sign; (4) find their difference and affix the sign of the greater sum.

Example. $(-30)-(-17)+(-6)-(+12)+(+2)$;

(1) $(-30)-(-17)+(-6)-(+12)+(+2)=-30+17-6-12+2$,

(2) $17+2=19$,

(3) $30+6+12=48$,

(4) $48-19=29$

The result is the negative number —29 since the greater sum (48) was obtained by combining the absolute values of those numbers which had minus signs in the expression $-30+17-6-12+2$.

This expression may be regarded both as a sum of the numbers -30, $+17$, -6, -12, $+2$ and as the result of the following successive operations: the addition of 17 to —30, the subtraction of 6, the subtraction of 12 and, finally, the addition of 2. Generally, the expression $a-b+c-d$ and so on may be regarded as the sum of the numbers $(+a)$, $(-b)$, $(+c)$, $(-d)$ and also as the result of the following successive operations: the subtraction of $(+b)$ from $(+a)$, the addition of $(+c)$, the subtraction of $(+d)$, and so on.

3. **Multiplication.** To multiply two numbers, multiply their absolute values and to the product affix a plus sign if the signs of the factors are the same, and a minus sign if they are different.

Scheme (rule of signs in multiplication):

$$\boxed{\begin{array}{ccccc} + & \cdot & + & = & + \\ + & \cdot & - & = & - \\ - & \cdot & + & = & - \\ - & \cdot & - & = & + \end{array}}$$

Examples. $(+2.4)\cdot(-5)=-12$, $(-2.4)\cdot(-5)=12$, $(-8.2)\times(+2)=-16.4$.

When multiplying several factors together, the sign of the product is positive if the number of negative factors is even, and negative if the number of negative factors is odd.

Example.

$$\left(+\frac{1}{3}\right)\cdot(+2)\cdot(-6)\cdot(-7)\cdot\left(-\frac{1}{2}\right)=-14$$

(three negative factors)

$$\left(-\frac{1}{3}\right)\cdot(+2)\cdot(-3)\cdot(+7)\cdot\left(+\frac{1}{2}\right)=7$$

(two negative factors)

4. **Division.** To divide one number by another, divide the absolute value of the former by the absolute value of the latter and place a plus sign in front of the quotient if the signs of the dividend and divisor are the same, and

a minus sign if they are different (the scheme is the same as for multiplication).

Examples. $(-6):(+3)=-2$, $(+8):(-2)=-4$, $(-12):(-12)=+1$.

70. Operations with Monomials. Addition and Subtraction of Polynomials

A *monomial* is a product of two or more factors each of which is either a number, a letter or a power of a letter. For example, $2d$, a^3b, $3abc$, $-4x^2y^3$ are monomials. A single number or a single letter may also be regarded as a monomial.

Any one of the factors of a monomial may be taken as the coefficient of the monomial. The numerical factor (say, -4 in the expression $-4x^2yz^3$) is often taken as the coefficient. By isolating one of the factors as the coefficient, we simply wish to stress that the monomial is the result of multiplying the rest of the expression by the coefficient. By separating out the numerical factor as the coefficient, we wish to emphasize that the main role is played by the literal expression, which is repeated as an addend as many times as indicated by the numerical factor (coefficient).

Monomials are called similar if they are the same or if they differ solely in the coefficients. It is clear then that two monomials may be similar or dissimilar depending on what coefficients are taken. If the numerical factors are the coefficients, then like (or similar) monomials are those whose letter parts are the same. For example, the monomials ax^2y^2, bx^2y^2, cx^2y^2 are similar if the coefficients are taken to be a, b, c; the monomials $3x^2y^2$, $-5x^2y^2$, $6x^2y^2$ are similar if the coefficients are the numerical factors.

Addition of monomials. Generally speaking, the addition of two or more monomials can only be indicated; until we assign numbers to the letters, the sum of several terms cannot, as a rule, be simplified. The only simplification possible is in the case of similar terms; here, the sum of similar terms is a term whose coefficient is equal to the sum of their coefficients. This procedure is called collecting like terms.

Example 1. $3x^2y^2-5x^2y^2+6x^2y^2=4x^2y^2$.

Example 2. $ax^2y^2-bx^2y^2+cx^2y^2=(a-b+c)\,x^2y^2$.

Example 3. $4x^3y^2-3x^2y^2-2x^3y^2+6x^2y^2+5xy$

$$=2x^3y^2+3x^2y^2+5xy.$$

Taking out a common factor. The operation performed in Example 2 is called factoring by taking out a common factor. We say that x^2y^2 has been factored out. Essentially, taking out a common factor is the same as collecting like terms.

Polynomials. The sum of any number of monomials is called a *polynomial*. The addition of two or more polynomials is nothing other than the formation of a new polynomial that includes all the terms of all the original polynomials.

Subtracting one polynomial from another is the same as adding a polynomial whose terms are those of the original polynomial with signs reversed.

Example. $(4a^2+2b-2x^2y^2)-(12a^2-c)+(7b-2x^2y^2)$
$=\underline{4a^2}+\underline{\underline{2b}}-\underline{\underline{\underline{2x^2y^2}}}-\underline{12a^2}+c+\underline{\underline{7b}}-\underline{\underline{\underline{2x^2y^2}}}=-8a^2+9b$
$-4x^2y^2+c$
(like terms are underlined with the same number of lines).

Multiplication of monomials. Generally speaking, the multiplication of monomials can only be indicated (cf. the foregoing on the addition of monomials). A product of two or more monomials can be simplified only when they include numerical coefficients or powers of the same letters, in which case the exponents of the appropriate letters are combined and the numerical coefficients multiplied together.

Example. $5ax^2y^5\,(-3a^3x^4z)=-15a^4x^6y^5z$.
[we add the exponents of the letter $a\,(1+3=4)$ and of the letter $x\,(2+4=6)$].

Division of monomials. Generally speaking, the division of one monomial by another one can only be indicated. The quotient of two monomials can be simplified if the dividend and divisor contain numerical coefficients or certain powers of the same letters, in which case the exponent of the divisor is subtracted from the exponent of the dividend, and the numerical coefficient of the dividend is divided by the numerical coefficient of the divisor.

Example. $12x^3y^4z^5:4x^2yz^2=3xy^3z^3$ [the exponents of the letter x are subtracted $(3-2=1)$, the same for y $(4-1=3)$ and for z $(5-2=3)$].

Note 1. If the exponents of some letter are the same in the dividend and in the divisor, the letter is dropped in the quotient because, divided by itself, it yields unity. Subtracting exponents we would get 0. We therefore agree to consider the zeroth power of any number to be the number 1.

Example. $4x^2y^3:2x^2y=2x^0y^2=2y^2\;(x^0=1)$.

Note 2. If the exponent of some letter in the dividend is less than the exponent of the same letter in the divisor, subtraction yields a negative power of that letter. For details concerning negative powers, see Sec. 125. The result can also be given in the form of a fraction. Then the negative exponent is not present.

Example. $10x^2y^5:2x^6y^4=5x^{-4}y=\frac{5y}{x^4}$ because $x^{-4}=\frac{1}{x^4}$.

71. Multiplication of Sums and Polynomials

The product of a sum of two or more expressions by an expression is equal to the sum of the products of each of the terms by that expression:

$(a+b+c)\,x=ax+bx+cx$ (with brackets removed)

In place of the letters a, b, c we could take any expressions, say, any monomials. The letter x can also be replaced by any expression; if that expression is itself a sum of several terms, say $m+n$, then we have $(a+b+c)(m+n)=$ $=a(m+n)+b(m+n)+c(m+n)=am+an+bm+bn+cm+cn$. That is, the product of a sum by a sum is equal to the sum of all possible products of each term of one sum by each term of the other sum.

For example, this rule refers to the product of a polynomial by a polynomial:

$$\begin{aligned}(3x^2-2x+5)(4x+2)&=12x^3-8x^2+20x+6x^2-4x+10\\&=12x^3-2x^2+16x+10\end{aligned}$$

Outline for multiplication:

$$\begin{array}{r}3x^2-2x+5\phantom{{}+10}\\4x+2\phantom{{}+10}\\\hline 12x^3-8x^2+20x\phantom{{}+10}\\6x^2-4x+10\\\hline 12x^3-2x^2+16x+10\end{array}$$

72. Formulas for Short-Cut Multiplication of Polynomials

The following special cases of multiplication of polynomials are frequently encountered and so should be memorized. It is particularly important to get into the habit of using these formulas for cases when the letters a and b are replaced by more complicated expressions (say, monomials).

1. $(a+b)^2=a^2+2ab+b^2$. The square of the sum of two quantities is equal to the square of the first term, plus twice the product of the two terms, plus the square of the second term.

Example 1. $104^2=(100+4)^2=10{,}000+800+16=10{,}816$.
Example 2. $(2ma^2+0.1nb^2)^2=4m^2a^4+0.4mna^2b^2+0.01n^2b^4$.
Warning: $(a+b)^2$ is not equal to a^2+b^2.

2. $(a-b)^2=a^2-2ab+b^2$. The square of the difference between two quantities is equal to the square of the first term, minus twice the product of the two terms, plus the square of the second term. This formula may be regarded as a special case of the preceding one [in place of b we take $(-b)$].

Example 1. $98^2=(100-2)^2=10{,}000-400+4=9604$.
Example 2. $(5x^3-2y^3)^2=25x^6-20x^3y^3+4y^6$.
Warning: $(a-b)^2$ is not equal to a^2-b^2 [see 3].

3. $(a+b)(a-b)=a^2-b^2$. The product of the sum and difference of two quantities is equal to the difference of their squares.

Example 1. $71\cdot69=(70+1)(70-1)=70^2-1=4899$.
Example 2. $(0.2a^2b+c^3)(0.2a^2b-c^3)=0.04a^4b^2-c^6$.

4. $(a+b)^3=a^3+3a^2b+3ab^2+b^3$. The cube of the sum of two quantities is equal to the cube of the first term, plus three times the product of the square of the first term by the second, plus three times the product of the first term by the square of the second, plus the cube of the second.

Example 1. $12^3=(10+2)^3=10^3+3\cdot10^2\cdot2+3\cdot10\cdot2^2$
$+2^3=1728$.

Example 2. $(5ab^2+2a^3)^3=125a^3b^6+150a^5b^4+60a^7b^2+8a^9$.
Warning: $(a+b)^3$ is not equal to a^3+b^3 [see 6].

5. $(a-b)^3=a^3-3a^2b+3ab^2-b^3$. The cube of the difference of two quantities is equal to the cube of the first term, minus three times the product of the square of the first term by the second, plus three times the product of the first term by the square of the second, minus the cube of the second term.

Example. $99^3=(100-1)^3=1{,}000{,}000-3\cdot10{,}000\cdot1+3$
$\times100\cdot1-1=970{,}299$.

Warning: $(a-b)^3$ is not equal to a^3-b^3 [see 7].

6. $(a+b)(a^2-ab+b^2)=a^3+b^3$. The product of the sum of two quantities by the "imperfect square of the difference" is equal to the sum of their cubes.

7. $(a-b)(a^2+ab+b^2)=a^3-b^3$. The product of the difference of two quantities by the "imperfect square of the sum" is equal to the difference of their cubes.

73. Division of Sums and Polynomials

The quotient obtained by dividing the sum of two or more expressions by an expression is equal to the sum of the quotients obtained by dividing each term by that expression:

$$\frac{a+b+c}{x}=\frac{a}{x}+\frac{b}{x}+\frac{c}{x}$$

a, b, c, x are any expressions; if they are all monomials, that is, if division of a polynomial by a monomial is performed, then each of the quotients may be simplified (see Sec. 70).

Example. $\frac{3a^2b+11ab^2}{ab}=\frac{3a^2b}{ab}+\frac{11ab^2}{ab}=3a+11b$.

If a, b, c are monomials and x is a polynomial, that is if division of a polynomial by a polynomial is performed, the quotient cannot, generally speaking, be represented as a polynomial (just as the quotient obtained by dividing a whole number by a whole number cannot always be represented as a whole number). To put it differently, we cannot always find a polynomial which, when multiplied by the polynomial of the divisor, will yield the polynomial of the dividend.

Example. The quotient $\frac{a^2+x^2}{a+x}$ cannot be represented as a polynomial; the quotient $\frac{a^2-x^2}{a+x}$ can be represented as a polynomial: $\frac{a^2-x^2}{a+x}=a-x$.

In the general case, the division of a polynomial by a polynomial can be carried out so that a remainder is left over, just as in the case of the division of whole numbers. However, it is necessary to establish just what is meant by the division of polynomials involving a remainder. If we divide the positive integer 35 by the positive integer 4, we get 8 and a remainder of 3. The numbers 8 and 3 have the property that $4\cdot8+3=35$, that is if p is the dividend, q the divisor, m the quotient and n the remainder, then $mq+n=p$. But this does not suffice to determine the quotient and remainder completely; thus, in our example ($p=35$, $q=4$) the numbers $m=6$, $n=11$, $m=4$, and $n=19$

have the same property. Also note that the number n must be less than q. This cannot be literally carried over to the case of division of polynomials because the same expression may be greater for one set of values of the letters and smaller than the other for another set. A modification is required: in each of the polynomials, some one of the letters of its terms is taken as the principal letter. The highest power of this letter is called the *degree of the polynomial*. Then division in which a remainder occurs is defined as follows.

To divide a polynomial P by a polynomial Q means to find a polynomial M (quotient) and a polynomial N (remainder) that satisfy two requirements: (1) the equality $MQ+N=P$ must hold, (2) the degree of the polynomial N must be lower than that of the polynomial Q.

Note. The remainder N need not contain the principal letter at all; then we say that N is of degree zero.

It is always possible to find polynomials M and N which satisfy these requirements uniquely for a given choice of the principal letter. However, they may differ if the choice of principal letter is different. The process of finding the quotient M and the remainder N is similar to the process of division (with a remainder) of one multidigit number by another. The role of higher- and lower-order digits is played by terms containing the principal letter to higher and lower powers. Before beginning the division process, the terms of the dividend and divisor are arranged in a descending order of powers of the principal letter.

Outline of division:

$$\begin{array}{r|l}
\text{(dividend) } 8a^3+16a^2-2a+4 & \underline{4a^2-2a+1 \text{ (divisor)}} \\
\underline{\mp 8a^3 \pm 4a^2 \mp 2a \quad\;} & \quad\quad 2a+5 \text{ (quotient)} \\
-\;20a^2-\;\;4a+4 & \\
\underline{20a^2 \pm 10a \mp 5} & \\
6a-1 & \text{(remainder)}
\end{array}$$

(1) Divide the first term of the dividend $8a^3$ by the first term of the divisor $4a^2$; the result $2a$ is the first term of the quotient.

(2) Multiply this term by the divisor $4a^2-2a+1$ to get $8a^3-4a^2+2a$ which is written under the dividend, similar terms under each other.

(3) Subtract the terms of the result from the corresponding terms of the dividend; bring down the next term of the dividend to get $20a^2-4a+4$.

(4) Divide the first term of the remainder $20a^2$ by the first term of the divisor to get 5, which is the second term of the quotient.

(5) Multiply the second term of the quotient by the divisor to get $20a^2-10a+5$, which is written under the first remainder.

(6) Subtract the terms of this result from the corresponding terms of the first remainder to get the second remainder $6a-1$. Its degree is less than the degree of the divisor, which means the division process is complete. We have a quotient of $2a+5$ and a remainder of $6a-1$.

Alternation mode of division:

$$\begin{array}{lrl}
 & 2x^2+3x^2-2x-4 & \text{(quotient)} \\
\cline{2-2}
\text{(divisor) } 3x-2 & \big|\ 6x^4+5x^3-12x^2-8x+3 & \text{(dividend)} \\
 & 6x^4-4x^3 & \\
\cline{2-2}
 & +9x^3-12x^2 & \\
 & 9x^3-\ 6x^2 & \\
\cline{2-2}
 & -6x^2-8x & \\
 & -6x^2+4x & \\
\cline{2-2}
 & -12x+3 & \\
 & -12x+8 & \\
\cline{2-2}
 & -5 & \text{(remainder)}
\end{array}$$

74. Division of a Polynomial by a First-Degree Binomial

If a polynomial containing x is divided by a first-degree binomial $x-l$, where l is some number (positive or negative), then we get a remainder which can only be a zero-degree polynomial (see Sec. 73), which is to say, some number N. The number N may be sought without finding the quotient. Namely, this number is equal to that value of the dividend which the latter assumes when $x=l$.

Example 1. Find the remainder left after dividing the polynomial x^3-3x^2+5x-1 by $x-2$. Substituting $x=2$ into the given polynomial, we find $N=2^3-3\cdot2^2+5\cdot2-1=5$.

Indeed, performing the division, we find the quotient $M=x^2-x+3$ and a remainder $N=5$.

Example 2. Find the remainder obtained from dividing the polynomial x^4+7 by $x+2$. Here, $l=-2$. Substituting $x=-2$ into x^4+7, we find $N=(-2)^4+7=23$.

This property of a remainder is called the *remainder theorem*. It was discovered by the French mathematician

Bézout (1730—1783). The remainder theorem reads: when a polynomial

$$a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \ldots + a_m$$

is divided by $x - l$, the remainder obtained is

$$N = a_0l^m + a_1l^{m-1} + a_2l^{m-2} + \ldots + a_m$$

Proof. By the definition of division (Sec. 73), we have

$$a_0x^m + a_1x^{m-1} + \ldots + a_m = (x - l)\,Q + N$$

where Q is a polynomial and N is a number. Substitute $x = l$; the term $(x - l)\,Q$ vanishes and we get

$$a_0l^m + a_1l^{m-1} + \ldots + a_m = N$$

Note. It may happen that $N = 0$. Then l is a root of the equation

$$a_0x^m + a_1x^{m-1} + \ldots + a_m = 0 \tag{1}$$

Example. The polynomial $x^3 + 5x^2 - 18$ leaves no remainder when divided by $x + 3$. The quotient is $x^2 + 2x - 6$. Hence, -3 is a root of the equation $x^3 + 5x^2 - 18 = 0$. Indeed, $(-3)^3 + 5(-3)^2 - 18 = 0$.

Conversely, if l is a root of equation (1), then the left member of this equation is exactly divisible by $x - l$.

Example. The number 2 is a root of the equation $x^3 - 3x - 2 = 0$ $(2^3 - 3\cdot 2 - 2 = 0)$. Hence, the polynomial $x^3 - 3x - 2$ is exactly divisible by $x - 2$. Indeed,

$$(x^3 - 3x - 2):(x - 2) = x^2 + 2x + 1$$

75. Divisibility of the Binomial $x^m \mp a^m$ by $x \mp a$

1. The difference between identical powers of two numbers is exactly divisible by the difference between the numbers, that is $x^m - a^m$ is divisible by $x - a$. This criterion, like the ones which follow, are a consequence of the remainder theorem (Sec. 74).

The quotient consists of m terms and has the following form: $(x^m - a^m):(x - a) = x^{m-1} + ax^{m-2} + a^2x^{m-3} + \ldots + a^{m-1}$ (the exponents on x steadily diminish by unity, whereas the exponents on a increase by unity so that the sum of the exponents is constant and equal to $m - 1$; all coefficients are equal to $+1$).

Examples.

$$(x^2-a^2):(x-a)=x+a,$$
$$(x^3-a^3):(x-a)=x^2+ax+a^2,$$
$$(x^4-a^4):(x-a)=x^3+ax^2+a^2x+a^3,$$
$$(x^5-a^5):(x-a)=x^4+ax^3+a^2x^2+a^3x+a^4.$$

2. The difference between identical even powers of two numbers is divisible not only by the difference between these numbers (Item 1) but also by their sum, i. e., x^m-a^m, for even m, is divisible both by $x-a$ and by $x+a$. In the latter instance, the quotient is of the form $x^{m-1}-ax^{m-2}+a^2x^{m-3}-\ldots$ (the plus and minus signs alternate).

Examples.

$$(x^2-a^2):(x+a)=x-a,$$
$$(x^4-a^4):(x+a)=x^3-ax^2+a^2x-a^3,$$
$$(x^6-a^6):(x+a)=x^5-ax^4+a^2x^3-a^3x^2+a^4x-a^5.$$

Note. Since the difference between even powers is divisible by $x-a$ and by $x+a$, it is also divisible by x^2-a^2.

Examples.

$$(x^4-a^4):(x^2-a^2)=x^2+a^2,$$
$$(x^6-a^6):(x^2-a^2)=x^4+a^2x^2+a^4,$$
$$(x^8-a^8):(x^2-a^2)=x^6+a^2x^4+a^4x^2+a^6.$$

The law of formation of quotients is obvious; it is readily subsumed under the law of Item 1, for example,

$$(x^8-a^8):(x^2-a^2)=[(x^2)^4-(a^2)^4]:(x^2-a^2)$$
$$=(x^2)^3+a^2(x^2)^2+(a^2)^2x^2+(a^2)^3$$

2a.. The difference between identical odd powers of two numbers is not divisible by the sum of the numbers.

For example, neither x^3-a^3 nor x^5-a^5 is divisible by $x+a$.

3. The sum of the same powers of two numbers is never divisible by the difference between these numbers.

For example, $x-a$ does not divide into x^2+a^2 or x^3+a^3 or x^4+a^4.

4. The sum of the same odd powers of two numbers is divisible by the sum of the numbers (in the quotient, the plus and minus signs alternate).

Examples.

$$(x^3+a^3):(x+a)=x^2-ax+a^2,$$
$$(x^5+a^5):(x+a)=x^4-ax^3+a^2x^2-a^3x+a^4.$$

4a. The sums of identical even powers of two numbers are not divisible by their difference (Item 3) or even by the sum of the numbers. For example, x^2+a^2 is not divisible either by $x-a$ or by $x+a$.

76. Factorization of Polynomials

A polynomial can sometimes be represented in the form of a product of two or more polynomials. This is by far not always possible but when it is possible, it is often very difficult to find the required factors. Its practical utility lies in the fact that it permits simplifying expressions (say, when common factors can be found in the numerator and denominator of a fraction; for examples, see the next section). The following are some elementary cases of the factorization of polynomials.

1. If all terms of a polynomial contain the same expression as a factor it can be taken outside the brackets (see Sec. 70, addition of monomials).

Example 1. $7a^2xy-14a^5x^3=7a^2x\,(y-2a^3x^2)$.

Example 2. $6x^2y^3-2uxy^2+4u^2xy=2xy\,(3xy^2-uy+2u^2)$.

2. It is sometimes possible to break the terms up into several groups, take out a factor in each group, and find the same expression in all sets of brackets. This expression can then be factored out; factorization of the polynomial is complete.

Example 1. $ax+bx+ay+by=x\,(a+b)+y\,(a+b)$ $=(a+b)\,(x+y)$.

Example 2. $\underline{10a^3}-\underline{\underline{6b^3}}+\underline{\underline{4ab^2}}-\underline{15a^2b}$

$$=5a^2\,(2a-3b)+2b^2\,(2a-3b)=(2a-3b)\,(5a^2+2b^2).$$

Note. It is worth remembering that the expression $a-b$ can always be given in the form $-(b-a)$ so that what appears at first glance to be different expressions can easily be turned into the same expressions.

Example 3. $6ax-2bx+9by-27ay$

$$=2x\,(3a-b)+9y\,(b-3a)=2x\,(3a-b)-9y\,(3a-b)$$
$$=(3a-b)\,(2x-9y).$$

3. The transformation explained in Item 2 can sometimes be carried out after first introducing new terms that mutually cancel or after breaking up one of the terms into two summands.

Example 1. $a^2-x^2=a^2+ax-ax-x^2$

$$=a(a+x)-x(a+x)=(a+x)(a-x)$$

(cf. formula 3 of Sec. 72).

Example 2. $p^2+pq-2q^2=p^2+2pq-pq-2q^2$

$$=p(p+2q)-q(p+2q)=(p+2q)(p-q).$$

4. The technique of Item 3 can sometimes be circumvented by using some of the factorization formulas obtained by manipulating the formulas of short-cut multiplication (Sec. 72), namely:

$$a^2+2ab+b^2=(a+b)^2; \quad a^2-2ab+b^2=(a-b)^2;$$
$$a^2-b^2=(a+b)(a-b), \text{ etc.}$$

Example. $4x^2+20xy+25y^2$. Using the first of these formulas ($a=2x$, $b=5y$), we get

$$4x^2+20xy+25y^2=(2x+5y)^2$$

Successful factorization of polynomials into the largest possible number of factors depends largely on one's dexterity in handling and combining the techniques enumerated above.

Example. $12+x^3-4x-3x^2=12-3x^2+x^3-4x$
$=3(4-x^2)-x(4-x^2)=(4-x^2)(3-x)$
$=(2+x)(2-x)(3-x)$.

77. Algebraic Fractions

An *algebraic fraction* is an expression of the form $\frac{A}{B}$, where A and B denote any literal or numerical expressions, and the horizontal bar is the symbol for division. The dividend A is called the numerator, the divisor B is called the denominator. The fractions studied in arithmetic are a special case of algebraic fractions (in which the numerator and denominator are positive whole numbers). The rules for handling algebraic fractions are the same as those for the fractions of arithmetic (see Secs. 30-36). We therefore confine ourselves to a number of illustrative examples.

Reduction of Fractions

Example 1. The fraction $\frac{15a^2x^4}{21a^5x^3}$ may be reduced by dividing through by $3a^2x^3$; $\frac{15a^2x^4}{21a^5x^3}=\frac{5x}{7a^3}$.

Example 2. The fraction $\frac{2a^2-ab-3b^2}{2a^2-5ab+3b^2}$ may be reduced by cancelling out $2a-3b$. To see this, factor the numerator and the denominator (see Sec. 76, Item 3):

$$\frac{2a^2-ab-3b^2}{2a^2-5ab+3b^2}=\frac{(2a-3b)(a+b)}{(2a-3b)(a-b)}=\frac{a+b}{a-b}$$

Addition and Subtraction of Fractions

Example 1. To add the fractions $\frac{m}{a^2b}+\frac{n}{ab^2}$ take a^2b^2 for the common denominator; the additional factors are b for the first summand and a for the second:

$$\frac{m}{a^2b}+\frac{n}{ab^2}=\frac{mb+na}{a^2b^2}$$

Example 2.

$$\frac{a-b}{2a^2-ab-3b^2}-\frac{a+b}{2a^2-5ab+3b^2}=\frac{a-b}{(2a-3b)(a+b)}-\frac{a+b}{(2a-3b)(a-b)}$$

$$=\frac{(a-b)^2-(a+b)^2}{(2a-3b)(a+b)(a-b)}$$

$$=\frac{-4ab}{(2a-3b)(a^2-b^2)}.$$

Note. Only in exceptional cases do multi-termed denominators have common factors, and if there are common factors, it usually requires a good deal of time to find them. Searching for such factors is a good exercise in developing algebraic habits and so the continued attention devoted to this work in textbooks is quite justified. However, their practical utility is slight and it is very often much better simply to take the product of the given denominators as the common denominator and not search for the simplest one.

Multiplication and Division of Fractions

Example 1. $\frac{4a^2b}{3c^2d}\cdot\frac{2c^3d^2}{ab^3}=\frac{8acd}{3b^2}$. Simplification can be carried out either prior to multiplication or afterwards.

Example 2. $\frac{x^2-a^2}{x^2-bx+cx-bc}:\frac{x^2-ax-cx+ac}{x^2-b^2}$

$$=\frac{(x^2-a^2)(x^2-b^2)}{(x-b)(x+c)(x-a)(x-c)}=\frac{(x+a)(x+b)}{(x+c)(x-c)}=\frac{(x+a)(x+b)}{x^2-c^2}.$$

78. Proportions

The definitions of a ratio and a proportion are given in Sec. 62. From the proportion $\frac{a}{b}=\frac{c}{d}$ follows $ad=be$ (the product of the means is equal to the product of the extremes); conversely, from $ad=bc$ follow the proportions

$$\frac{a}{b}=\frac{c}{d}, \quad \frac{a}{c}=\frac{b}{d}, \quad \frac{d}{b}=\frac{c}{a}$$

and others. All these proportions may be obtained from the original one, $\frac{a}{b}=\frac{c}{d}$, using the following rules.

1. In the proportion $\frac{a}{b}=\frac{c}{d}$, the means may be interchanged, the extremes may be interchanged, or both may be interchanged at the same time. We have

$$\frac{a}{c}=\frac{b}{d}, \quad \frac{d}{b}=\frac{c}{a}, \quad \frac{d}{c}=\frac{b}{a}$$

2. In a proportion, the antecedent and the consequent of both ratios may be interchanged. From $\frac{a}{b}=\frac{c}{d}$ we get $\frac{b}{a}=\frac{d}{c}$. This proportion was obtained earlier (in the form of $\frac{d}{c}=\frac{b}{a}$). In the same way, nothing is altered by interchanging the antecedent and the consequent in the three proportions given above.

Derived proportions. If $\frac{a}{b}=\frac{c}{d}$, then the following proportions (called *derived proportions*) which are obtained from the given proportion hold true:

$$\frac{a+b}{a}=\frac{c+d}{c}, \quad \frac{a-b}{a}=\frac{c-d}{c}, \quad \frac{a+b}{b}=\frac{c+d}{d}, \quad \frac{a-b}{b}=\frac{c-d}{d},$$

$$\frac{a}{a+b}=\frac{c}{c+d}, \quad \frac{a}{a-b}=\frac{c}{c-d}, \quad \frac{b}{a+b}=\frac{d}{c+d}, \quad \frac{b}{a-b}=\frac{d}{c-d},$$

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}, \quad \frac{a+c}{b+d}=\frac{a}{b}=\frac{c}{d}, \quad \frac{a+b}{c+d}=\frac{a}{c}=\frac{b}{d},$$

$$\frac{a-b}{c-d}=\frac{a}{c}=\frac{b}{d}, \quad \frac{a-c}{b-d}=\frac{a}{b}=\frac{c}{d}$$

These and a multitude of other derived proportions can be combined into two basic forms:

$$\frac{ma+nb}{m_1a+n_1b}=\frac{mc+nd}{m_1c+n_1d}, \tag{1}$$

$$\frac{ma+nc}{m_1a+n_1c}=\frac{mb+nd}{m_1b+n_1d} \tag{2}$$

where m, n, m_1, n_1 are any numbers. Form (2) is obtained by the same rule as (1) if we first interchange the means in the given proportion.

Putting $m=n=m_1=1$, $n_1=0$ in formula (1), we get the derived proportion $\frac{a+b}{a}=\frac{c+d}{c}$; putting $m=n=m_1=1$, $n_1=0$ in (2), we have $\frac{a+c}{a}=\frac{b+d}{b}$ or, interchanging the mean terms, $\frac{a+c}{b+d}=\frac{a}{b}$, and so on.

79. Why We Need Equations

Computational problems are of two kinds: direct and indirect.

Here is an instance of a *direct problem:* what is the weight of a chunk of alloy which contains 0.6 dm³ of copper (specific weight: 8.9 kg/dm³) and 0.4 dm³ of zinc (specific weight 7.0 kg/dm³)? Solving, we find the weight of the copper ($8.9 \cdot 0.6 = 5.34$ (kg)), the weight of the zinc ($7.0 \cdot 0.4 = 2.8$ (kg)) and then the weight of the alloy ($5.34 + 2.8 = 8.14$ (kg)). These operations and their sequence are implied in the very statement of the problem.

Here is an instance of an *indirect problem:* an alloy consisting of copper and zinc of volume 1 dm³ weighs 8.14 kg. Find the volumes of the copper and zinc in this alloy. Here, the statement of the problem does not indicate the operations that will lead to the solution. In what is called an arithmetic solution, considerable ingenuity is often required to find a plan of solution of an indirect problem. Each new problem requires setting up a fresh scheme. This is largely a waste of the computor's time. It was to rationalize the computing process that the method of equations, which is the basic subject of study in algebra, was created (see Sec. 65). The gist of the method is this.

1. The desired quantities are given special designations. For this purpose we use literal symbols (mostly the last letters of the Latin alphabet, x, y, z, u, v). Using these symbols and the signs of operations (+, —, etc.), we translate the conditions of the problem into mathematical language, that is, we express the relationships between the given quantities and the unknown quantities by mathematical symbols instead of words. Each such mathematical statement is an equation.

2. The next step is to solve the equation, that is, to find the values of the sought-for unknown quantities. The solution of an equation is a mechanical procedure, which is in strict accord with established rules. At this stage we no

longer have to take into account the specific features of the problem at hand, all we have to do is make use of firmly set rules and techniques. (One of the primary tasks of algebra is to derive these rules.)

Thus, equations are needed to mechanize the labour of the computor. After the equation has been set up, its solution can be obtained quite automatically (which is exactly what computing machines do at present). The difficulty of solving a problem lies in setting up the equation.

80. How to Set Up Equations

To form an equation is to express in mathematical form a relationship between the given (known) quantities of the problem and the sought-for (unknown) quantities. This relationship is sometimes so explicitly stated in the formulation of the problem that setting up the equation actually reduces to a word-for-word rehashing of the problem in the language of mathematical symbolism.

Example 1. Petrov received 16 roubles more for a job than half the sum which Ivanov got. Together their pay came to 112 roubles. How much did each get?

Denote by x the share of Ivanov. Half his pay is $1/2x$; Petrov's pay for the month was $1/2x+16$; together they received a total of 112 roubles. In symbols we can write

$$(1/2x+16)+x=112$$

The equation has been set up. Solving it by the established rules (Sec. 84), we find that Ivanov got paid $x=64$ roubles; Petrov's pay was $1/2x+16=48$ roubles.

More often, however, it happens that the relationship between the known and unknown quantities is not explicitly stated in the problem and has to be built up out of the statement of the problem. Practical problems are almost all of that kind so that the case we gave above is of a rather artificial type hardly ever encountered in real situations.

It is therefore impossible to give exhaustive advice as to how equations are set up. However, as a starting piece of advice we offer the following. For the value of the unknown (or unknowns) take some number or numbers (at random) and make a check to see if the guess is a solution to the problem or not. If we can make the check and see if our guess is correct or incorrect (which is more likely), then we can immediately set about forming the equation (or equations) we need. Namely, write down the very operations that were used to check the correctness of the randomly chosen number by introducing instead a literal symbol for the unknown. That is the equation we need.

Example 2. A piece of an alloy composed of copper and zinc weighs 8.14 kg and is 1 dm^3 in volume. How much copper is there in the alloy (specific weights: copper 8.9 kg/dm^3, zinc, 7.0 kg/dm^3)?

Take some number to express the desired volume of copper, say 0.3 dm^3. Now check to see how close a fit we get. Since 1 dm^3 of copper weighs 8.9 kg, 0.3 dm^3 will weigh $8.9 \cdot 0.3 = 2.67$ (kg). The volume of zinc in the alloy is $1 - 0.3 = 0.7$ (dm^3). This weighs $7.0 \cdot 0.7 = 4.9$ (kg). The total weight of zinc and copper comes out to $2.67 + 4.9 = 7.57$ (kg). But the weight of our piece is given as 8.14 kg. Our guess was wide of the mark, but at least we have an equation whose solution will yield the correct answer. In place of the guess (0.3 dm^3), denote the volume of copper (in dm^3) by x. In place of the product $8.9 \cdot 0.3 = 2.67$ take the product $8.9x$. This is the weight of the copper in the alloy. In place of $1 - 0.3 = 0.7$ take $1 - x$; this is the volume of the zinc. In place of $7.0 \cdot 0.7 = 4.9$, take $7.0(1 - x)$, which is the weight of the zinc. In place of $2.67 + 4.9$, take $8.9x + 7.0(1 - x)$, which is the total weight of the zinc and the copper, and which is given as 8.14 kg. Thus $8.9x + 7.0 \times (1 - x) = 8.14$. The solution of this equation (see Sec. 79) yields $x = 0.6$. The solution can be checked in a variety of ways, which yield a variety of equations, all of which however will result in the same solution. Such equations are called equivalent equations (see Sec. 82).

Quite naturally, after setting up equations has become a firmly established habit, there is no need to make these preliminary checks of conjectured numbers; simply start out by designating the unknown quantity by some letter (x, y, etc.) and then proceed as if this letter (the unknown) were the number we wish to verify.

81. Essential Facts about Equations

Two expressions, numerical or literal, joined together by an equality sign (=) form an equation (numerical or literal).

Every true numerical equation, and also every literal equation which holds true for all numerical values of the letters in them is termed an *identity*.

Examples. (1) The numerical equation $5\cdot 3+1=20-4$ is an identity. (2) The literal equation $(a-b)(a+b)=a^2-b^2$ is an identity since for all numerical values of a and b the right and left sides yield the same number. A literal equation is one in which all or some of the known quantities are expressed by letters, otherwise the equation is termed *numerical*.

It must be pointed out specially which letters of the equation are to be taken as known and which are to be regarded as unknown quantities. The unknowns are ordinarily denoted by x, y, z, u, v, w. According to the number of unknowns, equations are called equations in one, two, three, etc. unknowns.

To solve a numerical equation means to find numerical values of the indicated unknowns which turn the equation into an identity. These values are called the *roots* of the equation.

To solve a literal equation means to find expressions of the unknowns in terms of the known quantities given in the equation, which, when substituted for the unknowns, turn the equation into an identity. The expressions thus found are called the *roots of the equation.*

Example 1. $\frac{2}{3+x}=\frac{1}{2}x$ is a numerical equation in one unknown, x. For $x=1$ the expressions $\frac{2}{3+x}$ and $\frac{1}{2}x$ form an identity that is, they yield one and the same number; $x=1$ is the root of the equation.

Example 2. $ax+b=cx+d$ is a literal equation in one unknown, x; when $x=\frac{d-b}{a-c}$ it becomes an identity, since the expressions $a\frac{d-b}{a-c}+b$ and $c\frac{d-b}{a-c}+d$ yield the same numbers for all values of the letters a, b, c, d $\left(\text{these expressions may be transformed to } \frac{ad-bc}{a-c}\right)$. The value $x=\frac{d-b}{a-c}$ is the root of the equation.

Example 3. $3x+4y=11$ is a numerical equation in two unknowns. When $x=1$, $y=2$ it becomes an identity: $3\cdot 1+4\cdot 2=11$. The values $x=1$, $y=2$ are roots of the equation; $x=-3$, $y=5$ are also roots of the equation. The values $x=2$, $y=1\frac{1}{4}$ are also roots. This equation has infinitely many roots but it is not an identity because, for example, when $x=2$ and $y=3$ the right and left members of the equation do not equal each other.

Example 4. $2x+3=2(x+1)$ is a numerical equation in one unknown. It does not become an identity for any values of x whatever (the right member may be represented as $2x+2$: no matter what $2x$ equals, adding 2 to $2x$ cannot yield the same number as adding 3 to $2x$). This equation does not have any roots.

82. Equivalent Equations. Solving Equations

Equivalent equations are equations that have the same roots; for instance, $x^2=3x-2$ and $x^2+2=3x$ are equivalent since they both have the roots $x=1$ and $x=2$. The process of solving an equation consists mainly in replacing a given equation by another one that is equivalent to the first.

The basic techniques involved in solving equations are the following.

1. Replacement of an expression by an equivalent one. For example, the equation

$$(x+1)^2=2x+5$$

may be replaced by the equivalent equation

$$x^2+2x+1=2x+5$$

2. The terms of an equation may be transposed (carried over from one member, or side, of an equation to the other); in this process the sign of each term is reversed. For instance in the equation $x^2+2x+1=2x+5$ we can transpose all terms to the left member, the terms $+2x$ and $+5$ go from the right member to the left member with the sign reversed. The equation $x^2+2x+1-2x-5=0$ or, what is the same thing, $x^2-4=0$, is equivalent to the original equation.

3. Both members of an equation can be multiplied or divided by one and the same expression. It is important here to bear in mind that the new equation may not be equivalent to the preceding equation if the expression used to multiply or divide has the property of becoming equal to zero.

Example. Given the equation $(x-1)(x+2)=4(x-1)$. Divide both sides by $x-1$ to get $x+2=4$. This equation has a unique root, $x=2$, yet the original equation has the root $x=2$ and also the root $x=1$. This root was "lost" in the division by $x-1$. On the contrary, when we multiply both members of the equation $x+2=4$ by $(x-1)$ a new root, $x=1$, appears in addition to the root $x=+2$. From this it does not in the least follow that we should not multiply or divide both members of an equation by an expression which is capable of becoming zero. The only thing we have to do, when performing such operations, is to note whether any old roots are lost or any new ones are introduced.

4. It is also possible to raise both members of an equation to the same power or extract the same root; however, the result may be equations that are not equivalent to the original one. For example, the equation $2x=6$ has one root, $x=3$; but the equation $(2x)^2=6^2$, or $4x^2=36$, has two roots: $x=3$ and $x=-3$. Before transforming an equation, check to see whether some of the original roots are lost or new ones introduced. Most important is to be sure no old roots are lost. The introduction of new roots is not so dangerous because any root can be substituted into the original equation and verified on the spot to see whether it satisfies the equation or not.

83. Equations Classified

An equation is called *algebraic* if each of its members (sides) is a polynomial or monomial (single term) (see Sec. 70) with respect to the unknown quantities.

Examples. $bx+ay^2=xy+2^m$ is an algebraic equation in two unknowns, but the equation $bx+ay^2=xy+2^x$ is *not algebraic* because the right member is not a polynomial in the letters x and y (the term 2^x is not a monomial in the letter x).

The degree of an algebraic equation. Transpose all terms of an algebraic equation to one side and collect like terms; if the equation contains only one unknown, then the degree of the equation is the greatest exponent on the unknown. If the equation contains several unknowns, then for each term of the equation it is necessary to form the sum of the exponents of all unknowns. The largest sum is termed the *degree of the equation*.

Example 1. The equation $4x^3+2x^2-17x=4x^3-8$ is a second-degree equation since we get $2x^2-17x+8=0$ when all terms are transposed to the left side.

Example 2. The equation $a^4x+b^5=c^5$ is an equation of the first degree since the highest degree of the unknown, x, is one.

Example 3. The equation $a^2x^5+bx^3y^3-a^8xy^4-2=0$ is an equation of the sixth degree since the sum of the exponents on the unknowns x and y comes to 5 for the first and third terms, 6 for the second and zero for the fourth; the largest sum is 6.

Equations whose solution reduces to the solution of an algebraic equation are sometimes also classed as algebraic. The degree of such an equation is the *degree of the algebraic equation to which it reduces*.

Example 4. $\frac{x+1}{x-1}=2x$ is an equation of the second degree although the second degree of the unknown is not evident. However, if it is replaced by an equivalent algebraic equation (by getting rid of the fraction), it becomes $2x^2-3x-1=0$. A first-degree equation in any number of unknowns is also called a *linear equation*.

84. First-Degree Equation in One Unknown

An equation of the first degree in one unknown may be transformed to one of the form $ax=b$, where a and b are given numbers or literal expressions containing known quantities. The solution (root) is of the form $x=\frac{b}{a}$. Manipulative difficulties occur only in carrying out transformations.

Example 1. $\frac{3x-5}{2(x+2)}=\frac{3x-1}{2x+5}-\frac{1}{x+2}$.

(1) Reduce the right member to a common denominator:

$$\frac{3x-5}{2(x+2)}=\frac{(3x-1)(x+2)-(2x+5)}{(2x+5)(x+2)}$$

(2) Remove brackets in the numerator of the right member and collect terms: $\frac{3x-5}{2(x+2)}=\frac{3x^2+3x-7}{(2x+5)(x+2)}$

(3) Multiply both members by $2(2x+5)(x+2)$ so as to clear the equation of fractions. (We leave the question as to whether or not extraneous roots have been introduced till after the solution.)

$$(3x-5)(2x+5)=2(3x^2+3x-7)$$

(4) Removing brackets, we get

$$6x^2+5x-25=6x^2+6x-14$$

(5) Transpose all unknowns to the left member, all known quantities to the right, and collect like terms. We finally get $-x=11$, the root of the equation is thus $x=-11$.

Substitute this value into the original equation and verify that this root is not an extraneous root.

Example 2. $\frac{x^2}{(x-a)(x-b)}+\frac{(x-a)^2}{x(x-b)}+\frac{(x-b)^2}{x(x-a)}=3.$

(1) Reduce the left member to a common denominator:

$$x(x-a)(x-b)$$

(The additional factors are: x for the first fraction, $x-a$ for the second, and $x-b$ for the third).

$$\frac{x^3+(x-a)^3+(x-b)^3}{x(x-a)(x-b)}=3$$

(2) We get rid of the fraction by multiplying both sides by $x(x-a)(x-b)$:

$$x^3+(x-a)^3+(x-b)^3=3x(x-a)(x-b)$$

(3) Removing brackets, we have

$$x^3+x^3-3ax^2+3a^2x-a^3+x^3-3bx^2+3b^2x-b^3$$
$$=3x^3-3ax^2-3bx^2+3abx$$

(4) Transpose unknown terms to the left and known quantities to the right member. Collecting like terms, we finally have

$$3a^2x-3abx+3b^2x=a^3+b^3 \text{ or } 3(a^2-ab+b^2)x=a^3+b^3$$

(5) We now find the root of the equation to be

$$x=\frac{a^3+b^3}{3(a^2-ab+b^2)}$$

This expression may be simplified by cancelling out a^2-ab+b^2:

$$x=\frac{a+b}{3}$$

85. A System of Two First-Degree Equations in Two Unknowns

After performing manipulations like those considered in the preceding section, an equation of the first degree in two unknowns x and y becomes $ax+by=c$, where a, b, c are given numbers or literal expressions.

Taken separately, such an equation has infinitely many roots. Assign arbitrary values to one of the unknowns (say x) and the value of y can be found from the equation in one unknown by substituting the value of x into our equation. For instance, in the equation $5x+3y=7$ we can put $x=2$; we then get $10+3y=7$, whence $y=-1$.

However, if the unknowns x and y are connected by two (not one) equations of the first degree, then they will have infinitely many solutions only in exceptional cases (see Sec. 87). Generally, a system of two first-degree equations in two unknowns has only one set of solutions (solution set). It may also happen (in exceptional cases again) that it does not have any solutions at all (see Sec. 87).

Solving a system of two first-degree equations in two unknowns may be reduced, in a variety of ways, to the solution of one equation of the first degree in one unknown. Two such procedures are discussed in the next section.

Problems that lead to a system of two equations in two unknowns can always be solved by means of one equation in one unknown; however, much attention must then be paid to computations which, when a system of equations is employed, are handled routinely in the very process of solving the system. The same goes for problems involving three or more unknowns. They may be solved with the aid of one or two unknown quantities. The larger the number of unknowns involved, the simpler (generally speaking) it is to set up each one of the equations, but the more difficult is the process of solution of the system. Therefore, practical considerations suggest introducing as few unknowns as possible but in such a way that setting up the equations does not become too involved.

Example. A piece of alloy made of copper and zinc weighs 8.14 kg and is 1 dm³ in volume. How much copper and zinc is there in the alloy (copper has a specific weight of 8.9 kg/dm³, zinc, 7.0 kg/dm³)? Denoting by x and y the unknown volumes of copper and zinc, we have two equations:

$$x+y=1, \tag{1}$$

$$8.9x+7.0y=8.14 \tag{2}$$

The first states that the total volume of copper and zinc (in dm³) is equal to 1; the second states that the total weight (in kg) is equal to 8.14 ($8.9x$ is the weight of the copper; $7.0y$, the weight of the zinc). Solving the system of equations (1)—(2) using the general rules of Sec. 86, we find

$x=0.6$, $y=0.4$. We solved this same problem in Sec. 80 (Example 2) using only one unknown, x. The suggestions given in Sec. 80 hold true as well for setting up a system of equations in two or more unknowns.

86. Solving a System of Two First-Degree Equations in Two Unknowns

(a) **Solving by the substitution method.** This method consists in the following: (1) using one equation, find an expression of one of the unknowns, say x, in terms of the known quantities and the other unknown, y; (2) substitute this expression into the second equation, which then contains only one unknown, y; (3) solve the equation and find the value of y; (4) substitute the value of y into the expression of the unknown x which was found at the beginning of the solution. This yields the value of x.

Example. Solve the system of equations

$$8x-3y=46,$$
$$5x+6y=13$$

(1) Use the first equation to find the expression of x in terms of the given numbers and the unknown y:

$$x=\frac{46+3y}{8}$$

(2) Substitute this expression into the second equation:

$$5\cdot\frac{46+3y}{8}+6y=13$$

(3) Solve the resulting equation:

$$5(46+3y)+48y=104,$$
$$230+15y+48y=104,$$
$$15y+48y=104-230,$$
$$63y=-126, \quad y=-2$$

(4) Substitute the value of y $(y=-2)$ into the expression $x=\frac{46+3y}{8}$. This yields $x=\frac{46-6}{8}=5$.

(b) **Solving by addition or subtraction.** This method consists in the following: (1) both members of one equation are multiplied by some factor; both members of the second equation are multiplied by another factor. These factors are chosen so that the coefficients of the unknowns in both equations have the same absolute value after the operation. (2) Add or subtract the equations depending on whether the signs of the equalized coefficients have the same or different signs; in this way one of the unknowns is eliminated. (3) Solve the resulting equation in one unknown. (4) The other unknown can be found in the same way, but it is ordinarily easier to substitute the value of the first unknown into any one of the given equations and solve the resulting equation in one unknown.

Example. Solve the system of equations

$$\begin{aligned} 8x - 3y &= 46, \\ 5x + 6y &= 13 \end{aligned}$$

(1) The easiest way is to equalize the absolute values of the coefficients of y; multiply both sides of the first equation by 2; and both sides of the second by 1 (which is to say, leave it unchanged):

$$\begin{array}{r|c|r} 8x - 3y = 46 & 2 & 16x - 6y = 92, \\ 5x + 6y = 13 & 1 & 5x + 6y = 13 \end{array}$$

(2) Add the two equations:

$$\begin{array}{rr} + & 16x - 6y = 92 \\ & 5x + 6y = 13 \\ \hline & 21x \qquad = 105 \end{array}$$

(3) Solve the resulting equation:

$$x = \frac{105}{21} = 5$$

(4) Substitute the value $x = 5$ into the first equation to get

$$40 - 3y = 46, \quad -3y = 46 - 40, \quad -3y = 6$$

Whence

$$y = \frac{6}{-3} = -2$$

The method of addition or subtraction is preferable to other methods: (1) when the absolute values of the coefficients of one of the unknowns are equal in the given equations (then the first of the steps in the solution process is not needed); (2) when it is seen at once that the numerical coefficients of one of the unknowns can be equalized by means of small integral factors; (3) when the coefficients of the equations contain literal expressions.

Example. Solve the system

$$(a+c)\,x-(a-c)\,y=2ab,$$
$$(a+b)\,x-(a-b)\,y=2ac$$

(1) Equalize the coefficients of x; multiply both sides of the first equation by $(a+b)$, of the second by $(a+c)$ to get

$$(a+c)(a+b)\,x-(a+b)(a-c)\,y=2ab\,(a+b),$$
$$(a+c)(a+b)\,x-(a-b)(a+c)\,y=2ac\,(a+c)$$

(2) Subtract the second equation from the first:

$$[(a-b)(a+c)-(a+b)(a-c)]\,y=2ab\,(a+b)-2ac\,(a+c)$$

(3) Solve the equation obtained:

$$y=\frac{2ab\,(a+b)-2ac\,(a+c)}{(a-b)(a+c)-(a+b)(a-c)}$$

This expression can be simplified considerably but the manipulations are involved. In the numerator and denominator, remove brackets, collect terms, factorize and cancel common factors to simplify the fraction:

$$\begin{aligned} y&=\frac{2a\,(ab+b^2-ac-c^2)}{(a^2-ab+ac-bc)-(a^2+ab-ac-bc)}\\ &=\frac{2a\,[(ab-ac)+(b^2-c^2)]}{-2ab+2ac}\\ &=\frac{2a\,[(b-c)\,a+(b-c)(b+c)]}{-2a\,(b-c)}\\ &=\frac{2a\,(b-c)(a+b+c)}{-2a\,(b-c)}=-(a+b+c)\end{aligned}$$

(4) To find x, equalize the coefficients of y in the original equations by multiplying the first by $(a-b)$ and the second by $(a-c)$. Subtracting one equation from the other, we solve the equation in one unknown and find

$$x=\frac{2ab\,(a-b)-2ac\,(a-c)}{(a-b)(a+c)-(a+b)(a-c)}$$

Performing the manipulations as in Item 3, we get $x=b+c-a$. Substituting of the value of y into one of the original equations would have required unwieldy computations; this happens very often when solving literal equations.

87. General Formulas for and Special Cases of Solving Systems of Two First-Degree Equations in Two Unknowns

The solution of a system of equations of the form

$$ax+by=c \tag{1}$$

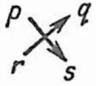
$$a_1x+b_1y=c_1 \tag{2}$$

may be obtained much faster if use is made of certain general formulas that have been worked out. These formulas may be found by any method, for example, by the method of addition and subtraction. The solution has the following form:

$$x=\frac{b_1c-bc_1}{ab_1-a_1b}\,, \tag{3}$$

$$y=\frac{ac_1-a_1c}{ab_1-a_1b} \tag{4}$$

These formulas are easy to remember if we introduce for numerators and denominators the following symbolism. We agree to use the symbol $\begin{vmatrix} p & q \\ r & s \end{vmatrix}$ to denote the expression $ps-rq$ which is obtained by cross multiplication:



and a subsequent subtraction of one product from the other (the product of the right-downwards diagonal has the plus sign). For example, the symbol $\begin{vmatrix} 5 & -8 \\ 2 & 1 \end{vmatrix}$ signifies $5\cdot1-2\times(-8)=5+16=21$.

The expression

$$\begin{vmatrix} p & q \\ r & s \end{vmatrix}=ps-rq$$

is termed a *determinant of the second order* (in contrast to determinants of orders three, four, etc. which arise in the solution of systems of first-degree equations in three, four, etc. unknowns).

With the aid of this symbolism, formulas (3) and (4) look like this:

$$x=\frac{\begin{vmatrix} c & b \\ c_1 & b_1 \end{vmatrix}}{\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix}} \quad (5), \qquad y=\frac{\begin{vmatrix} a & c \\ a_1 & c_1 \end{vmatrix}}{\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix}} \quad (6)$$

Each of the unknowns is equal to a fraction, the denominator of which is a determinant made up of the coefficients of the unknowns and the numerator is obtained from this determinant by replacing the coefficients of the corresponding unknown by the constant terms.

Example. Solve the system

$$8x-3y=46,$$
$$5x+6y=13$$

$$x=\frac{\begin{vmatrix} 46 & -3 \\ 13 & 6 \end{vmatrix}}{\begin{vmatrix} 8 & -3 \\ 5 & 6 \end{vmatrix}}=\frac{46\cdot 6+13\cdot 3}{8\cdot 6+5\cdot 3}=\frac{315}{63}=5,$$

$$y=\frac{\begin{vmatrix} 8 & 46 \\ 5 & 13 \end{vmatrix}}{\begin{vmatrix} 8 & -3 \\ 5 & 6 \end{vmatrix}}=\frac{8\cdot 13-5\cdot 46}{63}=\frac{-126}{63}=-2$$

An investigation shows that when solving the system (1)—(2), three essentially distinct cases arise.

(1) The coefficients of equations (1) and (2) are not proportional: $\frac{a}{a_1}\neq\frac{b}{b_1}$. Then, no matter what the constant terms, the equation has a unique solution given by formulas (3) and (4), or, what is the same, by formulas (5) and (6).

(2) The coefficients of (1) and (2) are proportional: $\frac{a}{a_1}=\frac{b}{b_1}$. Then it is important to know whether the constant terms as well are in that relation. If so, that is, if $\frac{a}{a_1}=\frac{b}{b_1}=\frac{c}{c_1}$, then the system of equations has an infinitude of solutions. The reason for this is that one of the equations is a consequence of the other, so that in actuality we are dealing with one equation and not two.

Example. In the system

$$10x+6y=18,$$
$$5x+3y=9$$

the coefficients of the unknowns x and y are proportional: $\frac{10}{5}=\frac{6}{3}=2$. The constant terms are in the same ratio $\frac{18}{9}=2$. Hence one of the equations is a consequence of the other; namely, the first is obtained from the second by multiplying both members of the latter by 2. Any one of the infinity of solutions of one of the equations is a solution of the other.

(3) The coefficients of the equations are proportional: $\frac{a}{a_1}=\frac{b}{b_1}$ but the constant terms are not in that ratio. Then the system has no solution because the equations are contradictory.

Example. In the system

$$\begin{aligned}10x+6y&=20,\\ 5x+3y&=9\end{aligned}$$

the coefficients are proportional: $\frac{10}{5}=\frac{6}{3}=2$. The ratio of the constant terms is different from the ratio of the coefficients: $\frac{20}{9}=2\frac{2}{9}$. The system has no solution because if we multiply the second equation by 2, we get $10x+6y=18$ which contradicts the first equation, for one and the same expression $10x+6y$ cannot be equal to 18 and to 20.

88. A System of Three First-Degree Equations in Three Unknowns

If transformations similar to those indicated in Sec. 84 are carried out, a first-degree equation in three unknowns x, y, z takes the form $ax+by+cz=d$, where a, b, c, d are given numbers or literal expressions. One such equation, taken separately, or a system of two such equations has an infinity of solutions. In the general case, a system of three first-degree equations in three unknowns has one set of solutions. In exceptional cases (see below) it can have infinitely many solutions or none at all.

The solution of a system of three equations in three unknowns is based on the same techniques as used in the solution of a system of two equations in two unknowns, as will be seen from the following example.

Example. Solve the system of equations

$$3x-2y+5z=7, \tag{1}$$
$$7x+4y-8z=3, \tag{2}$$
$$5x-3y-4z=-12 \tag{3}$$

Take two equations of this system, say (1) and (2), and assume that one of the unknowns (say z) has already been found, that is, is known. Solving this system for x and y by the rules of Sec. 86, we get

$$x=\frac{17-2z}{13}, \quad y=\frac{59z-40}{26} \tag{4}$$

Substituting these expressions of x and y into (3), we get an equation in one unknown:

$$\frac{5(17-2z)}{13}-\frac{3(59z-40)}{26}-4z=-12$$

Solving this equation (see Sec. 84), we obtain $z=2$. Putting this value in (4), we find $x=1$, $y=3$.

The general formulas for solving the system

$$\left.\begin{aligned} ax+by+cz&=d \\ a_1x+b_1y+c_1z&=d_1 \\ a_2x+b_2y+c_2z&=d_2 \end{aligned}\right\} \tag{5}$$

may be obtained by the same device. The solution will be complicated and hard to remember if written out in full, but it can be given a convenient and easily remembered form if we first introduce the concept of a *third-order determinant*.

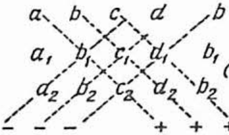
A determinant of third order, symbolized compactly as

$$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \tag{6}$$

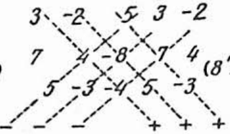
is simply the expression

$$ab_1c_2+bc_1a_2+ca_1b_2-cb_1a_2-ac_1b_2-ba_1c_2 \tag{7}$$

This expression need not be memorized since it is readily obtainable from its symbol (6) as follows: rewrite the array (6), adjoining on the right the first two columns. It then takes the form (8):



(8)



(8′)

Draw diagonal lines [shown in (8) by the dashed lines] and write out the products of the letters on each of the six diagonals. Affix the plus sign to the three products that represent the diagonals from upper left to lower right, and the minus sign to the other three products. Writing out these products horizontally, we get (7).

Example 1. Compute the third-order determinant

$$\begin{vmatrix} 3 & -2 & 5 \\ 7 & 4 & -8 \\ 5 & -3 & -4 \end{vmatrix} \tag{9}$$

Scheme (8) becomes scheme (8′).

The determinant (9) is then

$$3\cdot 4\,(-4)+(-2)\cdot(-8)\cdot 5+5\cdot 7\cdot(-3)-5\cdot 4\cdot 5-3\cdot(-8)\cdot(-3)$$
$$-(-2)\cdot 7\cdot(-4)=-48+80-105-100-72-56=-301$$

Using determinants, we can represent the system (5) as

$$x=\frac{\begin{vmatrix} d & b & c \\ d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}},\quad y=\frac{\begin{vmatrix} a & d & c \\ a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}\quad z=\frac{\begin{vmatrix} a & b & d \\ a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}} \tag{10}$$

Here, each of the unknowns is equal to a fraction: the denominator is a determinant made up of the coefficients of the unknowns, and the numerator is obtained from this determinant by replacing the coefficients of the corresponding unknown by the constant terms.

Example 2. Solve the system of equations

$$\begin{aligned} &3x-2y+5z=7,\\ &7x+4y-8z=3,\\ &5x-3y-4z=-12 \end{aligned}$$

The common denominator of the formulas (10) was computed in the example, it is −301. The numerator of the first of the formulas in (10) is obtained from (9) by replacing the first column by the column of constant terms. It looks like this:

$$\begin{vmatrix} 7 & -2 & 5 \\ 3 & 4 & -8 \\ -12 & -3 & -4 \end{vmatrix}$$

Computing it by (8) we get -301. Thus, we have $x=\frac{-301}{-301}=1$ (cf. the example on page 168). In the same way we find

$$y=\frac{-903}{-301}=3, \quad z=\frac{-602}{-301}=2$$

The system of equations (5) has a unique solution if the determinant made up of the coefficients of the unknowns is not equal to zero. Then formulas (10), with this determinant in the denominator, yield the solution of system (5). If the determinant made up of coefficients is zero, then formulas (10) become useless for the purpose of computation. In this case system (5) either has an infinity of solutions or none at all. There are an infinity of solutions if not only the determinant in the denominators but also the determinants in the numerators of formulas (10) vanish; it is important to note that if the determinant in the denominators and one of the determinants in the numerators are zero, then the other two determinants in the numerators must be zero. The existence of an infinity of solutions is due to the fact that one of the three equations (5) is a consequence of the other two [or even each of two of the equations (5) is a consequence of the third], so that actually we have not three but only two equations in three unknowns (or even one equation).

Example 3. In the system of equations

$$\left.\begin{aligned} 2x-5y+z&=-2\\ 4x+3y-6z&=1\\ 2x+21y-15z&=8 \end{aligned}\right\} \tag{11}$$

the determinant of coefficients is

$$\begin{vmatrix} 2 & -5 & 1\\ 4 & 3 & -6\\ 2 & 21 & -15 \end{vmatrix}=0$$

[see scheme (8)]. Taking one of the determinants in the numerators of formulas (10), say the determinant

$$\begin{vmatrix} -2 & -5 & 1\\ 1 & 3 & -6\\ 8 & 21 & -15 \end{vmatrix}$$

which is in the first one of (10) we find that it too is zero. The other two determinants in the second and third formulas of (10) need not be computed since they are certainly zero.

The system (11) has an infinite number of solutions; one of its equations (any one) is a consequence of the other two. For example, if we multiply the second equation by 2 and the first by —3 and combine the two equations, we get the third one.

The system (5) has no solutions at all if the determinant in the denominators of formulas (10) is zero but not one of the determinants in the numerators is zero. Here, it is sufficient to ascertain that one of the numerators is nonzero, then the other two will definitely be different from zero. The absence of solutions is due to the fact that one of the equations contradicts the other two (or even each one separately).

Example 4. Take the system of equations

$$\left.\begin{array}{l} 2x-5y+z=-2 \\ 4x+3y-6z=1 \\ 2x+21y-15z=3 \end{array}\right\} \qquad (12)$$

which differs from the system (11) solely in the value of the constant term in the last equation. The determinant of the coefficients therefore remains the same: equal to zero. However, the determinants in the numerators will be different. For instance, the numerator of the first of the formulas of (10) will be

$$\begin{vmatrix} -2 & -5 & 1 \\ 1 & 3 & -6 \\ 3 & 21 & -15 \end{vmatrix} = -135$$

It is not zero. The other two numerators are definitely not zero. The system (12) has no solutions. It is inconsistent because the first two equations yield $2x+21y-15z=8$ as a consequence (see Example 3); yet the third equation of (12) has the form $2x+21y-15z=3$. Thus one and the same expression is equal to both 3 and 8, which is impossible.

89. Laws of Exponents

1. The power of a product of two or more factors is equal to the product of the same power of each of the separate factors:

$$(abc\ldots)^n = a^n b^n c^n \ldots$$

Example 1. $(7\cdot 2\cdot 10)^2 = 7^2\cdot 2^2\cdot 10^2 = 49\cdot 4\cdot 100 = 19{,}600$.

Example 2. $(x^2-a^2)^3 = [(x+a)(x-a)]^3 = (x+a)^3(x-a)^3$ (cf. Sec. 72, Item 3).

Of more practical utility is the reverse transformation:

$$a^n b^n c^n \ldots = (abc\ldots)^n$$

where the product of the same powers of several quantities is equal to the same power of the product of the quantities.

Example 3. $4^3 \cdot \left(\frac{7}{4}\right)^3 \cdot \left(\frac{2}{7}\right)^3 = \left(4 \cdot \frac{7}{4} \cdot \frac{2}{7}\right)^3 = 2^3 = 8.$

Example 4. $(a+b)^2\ (a^2-ab+b^2)^2 = [(a+b)\ (a^2-ab+b^2)]^2 = (a^3+b^3)^2$ (cf. Sec. 72, Item 6).

2. The power of a quotient (or of a fraction) is equal to the quotient obtained by dividing the same power of the dividend by the same power of the divisor:

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Example 5. $\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4} = \frac{16}{81}.$

Example 6. $\left(\frac{a+b}{a-b}\right)^3 = \frac{(a+b)^3}{(a-b)^3}.$

The inverse transformation is: $\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n.$

Example 7. $\frac{7.5^3}{2.5^3} = \left(\frac{7.5}{2.5}\right)^3 = 3^3 = 27.$

Example 8. $\frac{(a^2-b^2)^2}{(a+b)^2} = \left(\frac{a^2-b^2}{a+b}\right)^2 = (a-b)^2$ (cf. Sec. 72, Item 3).

3. When multiplying powers having the same base, add the exponents (cf. Sec. 70):

$$a^m a^n = a^{m+n}$$

Example 9. $2^2 \cdot 2^5 = 2^{2+5} = 2^7 = 128.$

Example 10. $(a-4c+x)^2\,(a-4c+x)^3 = (a-4c+x)^5.$

4. In dividing powers having the same base, subtract the exponent of the divisor from the exponent of the dividend (cf. Sec. 70):

$$\frac{a^m}{a^n} = a^{m-n}$$

Example 11. $12^5 . 12^3 = 12^{5-3} = 12^2 = 144.$

Example 12. $(x-y)^3 : (x-y)^2 = x-y.$

5. In finding a power of a power, multiply the exponents: $(a^m)^n = a^{mn}$.

Example 13. $(2^3)^2 = 2^6 = 64.$

Example 14. $\left(\frac{a^2b^3}{c}\right)^4 = \frac{(a^2)^4\,(b^3)^4}{c^4} = \frac{a^8b^{12}}{c^4}.$

90. Operations Involving Radicals

In the formulas given below, the symbol $\sqrt{\ }$ denotes the absolute value of the root.

1. The value of a root remains unchanged if its index is increased n times and the radicand is raised to the nth power:

$$\sqrt[m]{a}=\sqrt[m\cdot n]{a^n}$$

Example 1. $\sqrt[3]{8}=\sqrt[3\cdot 2]{8^2}=\sqrt[6]{64}$.

2. The value of the root remains unchanged if its index is reduced n times and the nth root is taken of the radicand at the same time:

$$\sqrt[m]{a}=\sqrt[m:n]{\sqrt[n]{a}}$$

Example 2. $\sqrt[6]{8}=\sqrt[6:3]{\sqrt[3]{8}}=\sqrt{2}.$

Note. This property also holds true when the number $\frac{m}{n}$ is not integral; both properties are also valid when n is fractional. But to see this we first have to extend the concept of a power and a root by introducing fractional exponents (see Sec. 125).

3. The root of a product of several factors is equal to the product of the roots (of the same index) of these factors:

$$\sqrt[m]{abc\ \dots}=\sqrt[m]{a}\sqrt[m]{b}\sqrt[m]{c}\ \dots$$

Example 3. $\sqrt[3]{a^6b^2}=\sqrt[3]{a^6}\sqrt[3]{b^2}=a^2\sqrt[3]{b^2}.$

The last transformation is based on Property 2.

Example 4. $\sqrt{48}=\sqrt{16\cdot 3}=\sqrt{16}\cdot\sqrt{3}=4\cdot\sqrt{3}.$

Conversely, the product of roots of the same index is equal to the root (of that index) of the product of the radicands:

$$\sqrt[m]{a}\cdot\sqrt[m]{b}\cdot\sqrt[m]{c}\dots=\sqrt[m]{abc\ \dots}$$

Example 5. $\sqrt{a^3b}\cdot\sqrt{ab^3}=\sqrt{a^4b^4}=a^2b^2.$

4. The root of a quotient is equal to the quotient obtained by dividing the root of the dividend by the root of the

divisor (the indices of the roots are all assumed to be the same):

$$\sqrt[m]{a:b}=\sqrt[m]{a}:\sqrt[m]{b}$$

Conversely: $\sqrt[m]{a}:\sqrt[m]{b}=\sqrt[m]{a:b}$.

Example 6. $\sqrt[3]{27:4}=\sqrt[3]{27}:\sqrt[3]{4}=3:\sqrt[3]{4}$.

5. To raise a root to a power, raise the radicand to that power:

$$(\sqrt[m]{a})^n=\sqrt[m]{a^n}$$

Conversely, to extract the root of a power, raise the root of the base to that power:

$$\sqrt[m]{a^n}=(\sqrt[m]{a})^n$$

Example 7. $(\sqrt[3]{a^2b})^2=\sqrt[3]{a^4b^2}=\sqrt[3]{a^3\cdot ab^2}=a\sqrt[3]{ab^2}$.

Example 8. $\sqrt{27}=\sqrt{3^3}=(\sqrt{3})^3=(\sqrt{3})^2\cdot\sqrt{3}=3\sqrt{3}$.

6. Rationalizing the denominator or numerator of a fraction. Computation of fractional expressions involving radicals is often simplified by what is termed "rationalizing" the numerator or denominator. This process signifies eliminating the radicals in the numerator or denominator.

Example 9. Let it be required to compute $\frac{1}{\sqrt{7}-\sqrt{6}}$ to 0.01. If the operations are carried out in the indicated order, we have (1) $\sqrt{7}\approx 2.646$, (2) $\sqrt{6}\approx 2.499$, (3) $2.646-2.449=0.197$, (4) $\frac{1}{0.197}\approx 5.10$. Four operations were required to obtain the answer; what is more, to obtain digits correct to hunredths required computing the roots to thousandths, otherwise we would have only two significant digits in the denominator of the fraction $\frac{1}{\sqrt{7}-\sqrt{6}}$ and so we could not obtain three correct significant digits (see Sec. 56).

But if we first multiply the numerator and denominator of the fraction by $\sqrt{7}+\sqrt{6}$, we get

$$\frac{1}{\sqrt{7}-\sqrt{6}}=\frac{\sqrt{7}+\sqrt{6}}{(\sqrt{7})^2-(\sqrt{6})^2}=\frac{\sqrt{7}+\sqrt{6}}{1}$$

The computation now requires only three operations, and the root need only be evaluated to hundredths:

(1) $\sqrt{7} \approx 2.65$, (2) $\sqrt{6} \approx 2.45$, (3) $\sqrt{7}+\sqrt{6} \approx 5.10$.

Here are some typical illustrations.

Example 10. $\frac{\sqrt{7}}{\sqrt{5}}=\frac{\sqrt{7}\cdot\sqrt{5}}{\sqrt{5}\cdot\sqrt{5}}=\frac{\sqrt{35}}{5}$.

Example 11. $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a})^2-(\sqrt{b})^2}=\frac{a+2\sqrt{ab}+b}{a-b}$.

In these examples, we rationalized the denominator. In the two following examples we rationalize the numerator.

Example 12. $\frac{\sqrt{7}}{\sqrt{5}}=\frac{\sqrt{7}\cdot\sqrt{7}}{\sqrt{5}\cdot\sqrt{7}}=\frac{7}{\sqrt{35}}$.

Example 13. $\frac{\sqrt{35}-\sqrt{34}}{3}=\frac{\sqrt{35^2}-\sqrt{34^2}}{3\,(\sqrt{35}+\sqrt{34})}=\frac{1}{3\,(\sqrt{35}+\sqrt{34})}$.

The transformation in Example 12 is clearly unsuitable for computational purposes because evaluating the expression $\frac{7}{\sqrt{35}}$ requires division by a multidigit number, whereas the computation of $\frac{\sqrt{35}}{5}$ (see Example 10) requires division by a whole number. But the transformation in Example 13 is useful in that it permits computing the roots $\sqrt{35}$ and $\sqrt{34}$ to as many places as required in the answer, whereas in the original expression we would have had to extract the roots to a larger number of places (see Example 9). Thus, the general school practice of rationalizing the denominator every single time is a harmful formalistic tradition.

91. Irrational Numbers

The range of whole and fractional numbers is more than sufficient for purposes of mensuration (see Sec. 45). It is not, however, sufficient for the theory of mensuration.

For example, let it be required to determine *exactly* the length of the diagonal AC in the square $ABCD$ (Fig. 1), the side of which is 1 metre long. The area of the square $ACEF$ constructed on the diagonal is equal to twice the area of $ABCD$ (the triangle ACB is contained twice in $ABCD$ and four times in $ACEF$). Therefore, if x is the desired length of AC, then $x^2=2$. However, no whole number and no fraction can satisfy this equation.

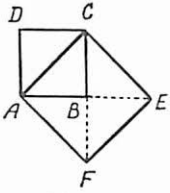


Fig. 1.

We can do one of two things: either give up attempting to express exactly lengths by numbers or introduce new numbers in addition to the whole and fractional numbers. After a long period of struggle, the latter point of view prevailed.

These new numbers, which represent the lengths of line segments that are incommensurable with the unit length (that is to say, line segments that cannot be expressed by any whole or fractional number) are called *irrational numbers*.* In contrast to irrational numbers, whole numbers (integers) and fractions were called *rational*. After the introduction of negative numbers (this was later; see Sec. 66), they too were split into rational and irrational.

Every rational number can be represented in the form $\frac{m}{n}$, where m and n are integers (positive or negative). Irrational numbers cannot be exactly represented in this form. However, every irrational number can be replaced approximately, to any degree of accuracy, by a rational number $\frac{m}{n}$; for instance, it is possible to find a decimal fraction (pure or mixed) which differs from a given irrational number by an arbitrarily small number.

The numbers $\sqrt{2}$, $\sqrt{5}$, $\sqrt[3]{3+\sqrt{2}}$, $\sqrt{\sqrt[3]{5}+\sqrt{7}}$ and many other expressions involving rational numbers under the radical sign are irrational. These irrational numbers are said to be expressed in terms of radicals.

They do not however exhaust the range of irrational numbers. Up to the end of the 18th century, mathematicians were convinced that the root of any algebraic equation with rational coefficients could be expressed in terms of radicals if the root is not rational; later it was proved that this is true only for equations up to the fourth degree inclusive (see Sec. 66). As a rule, the irrational roots of equations of fifth and higher degree cannot be expressed in terms of radicals. Numbers which are the roots of algebraic equations with integral coefficients are called *algebraic numbers;* only in exceptional cases are algebraic numbers expressible by radicals, and still rarer are the cases when they are rational.

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* The term "irrational" literally means "having no ratio". Originally, it did not refer to an irrational number but to quantities whose ratio is now expressed by an irrational number. Say, the ratio of the diagonal of a square to its side is now given by the number $\sqrt{2}$. Prior to the advent of irrational numbers, it was customary to say that the diagonals of a square cannot be related to its side.

Algebraic numbers still do not exhaust the range of irrational numbers. Thus, for instance, the well-known number π of geometry (see Sec. 152) is irrational, but it cannot be the root of any algebraic equation with integral coefficients. The same goes for the number e (see Sec. 128) which is not algebraic. In other words, pi (π) and e are not algebraic numbers.

An irrational number which cannot be the root of any algebraic equation with integral coefficients is called a *transcendental number*.

Up to 1929, only a few numbers had been proved to be transcendental; the transcendence of the number e was proved in 1871 by the French mathematician Hermite. In 1882 the German mathematician Lindemann proved the transcendence of π. Academician A. A. Markov (1856—1922) proved the transcendence of e and π by a new method. In 1913 D.D. Mordukhai-Boltovskoi (1877—1952) pointed out a number of new transcendental numbers. It was still not known, however, whether such "ordinary" numbers as $3^{\sqrt{2}}$, $\sqrt{3}^{\sqrt{2}}$ were transcendental or not. The Soviet mathematicians A. O. Gelfond and R. O. Kuzmin (1891—1949) proved in 1929 and 1930 that all numbers of the form $\alpha^{\sqrt{n}}$, where α is an algebraic number not equal to zero or unity and n is an integer, are transcendental. The numbers $3^{\sqrt{2}}$, $\sqrt{3}^{\sqrt{2}}$, etc., are precisely of this form. In 1934 Gelfond completed these studies. He proved the transcendence of all numbers of the form α^{β} where α and β are arbitrary algebraic numbers (provided that α is neither 0 nor 1 and β is irrational). For example, the number $\left(\sqrt[4]{5}\right)^{\sqrt[3]{2}}$ is transcendental. From the transcendence of the numbers α^{β} there readily follows the transcendence of the decimal logarithms of all integers (except, of course, 1, 10, 100, 1000, etc.).

92. Quadratic Equations. Imaginary and Complex Numbers

An algebraic equation of the second degree is called a *quadratic* equation. The most general form of a quadratic equation in one unknown is

$$ax^2+bx+c=0$$

where a, b, c are given numbers or literal expressions involving known quantities (note that the coefficient a cannot be zero for otherwise the equation would not be quadratic, but only of degree one). Dividing both members of the equation by a, we get an equation of the form

$$x^2+px+q=0 \quad \left(p=\frac{b}{a},\ q=\frac{c}{a}\right)$$

A quadratic equation of this kind is called *reduced*; the equation $ax^2+bx+c=0$ (where $a \neq 1$) is called *unreduced*. If one of the quantities b, c or both are zero, then the quadratic equation is termed *incomplete*; if both b and c are nonzero the quadratic equation is called *complete*.

Examples

$3x^2+8x-5=0$, complete unreduced quadratic equation
$3x^2-5=0$, incomplete unreduced quadratic equation
$x^2-ax=0$, incomplete reduced quadratic equation
$x^2-12x+7=0$, complete reduced quadratic equation.

An incomplete quadratic equation of the form

$$x^2=m \text{ } (m \text{ known})$$

is the simplest case of a quadratic equation and a very important case too, since the solution of every quadratic reduces to this form. The solution of this equation is

$$x=\sqrt{m}$$

Three cases are possible:

(1) If $m=0$, then $x=0$.

(2) If m is a positive number, then its square root $\sqrt{m}$ can have two values: one positive and one negative. Their absolute values are the same. For instance, the equation $x^2=9$ is satisfied by the value $x=+3$ and $x=-3$. In other words, x has two values: $+3$ and -3. This is frequently expressed by the double (plus-and-minus) sign in front of the radical: $x=\pm\sqrt{9}$. This notation means that the

expression $\sqrt{9}$ denotes the general absolute value of two roots; in our case, the number 3. The quantity $\sqrt{m}$ may be irrational (see Sec. 91). Note too that the number m itself may be an irrational number. For instance, let it be required to solve the equation

$$x^2 = \pi$$

(geometrically, this means finding the length of a side of a square equal to the area of a circle of radius 1). Its root is $x = \sqrt{\pi}$. See Sec. 58 on the extraction of square roots of numbers.

(3) If m is a negative number, then the equation $x^2 = m$ (say, $x^2 = -9$) cannot have any positive or negative root: this is obvious since either a positive or a negative number, when squared, yields a positive number. We can thus say that the equation $x^2 = -9$ has no solution, that is, the number $\sqrt{-9}$ does not exist.

Prior to the introduction of negative numbers we would have been equally justified in saying that the equation $2x + 6 = 4$ has no solutions. But after the introduction of negative numbers this equation became solvable. In the same way, the equation $x^2 = -9$ which does not have any solutions in the set of the positive and negative numbers becomes solvable if we introduce new quantities: the square roots of negative numbers. These quantities were first introduced by the Italian mathematician Cardano in the middle of the 16th century in connection with the solution of cubic equations (see Sec. 66). Cardano called these numbers "sophistic". Descartes (in the 1630's) suggested the name "imaginary numbers" which, most unfortunately, has persisted to this day. In contradistinction to the imaginary numbers, the earlier known numbers (positive and negative, including the irrational numbers) came to be called *real numbers*. The sum of a real number and an imaginary number constitutes a so-called *complex number*. This term was introduced by Gauss in 1831. For instance, $2 + \sqrt{-3}$ is a complex number. Complex numbers too are sometimes called imaginary numbers. Complex numbers are explained in detail in Secs. 34 et seq.

Having at our disposal imaginary numbers, we can say that the incomplete quadratic equation $x^2 = m$ always has two roots. If $m > 0$, these roots are real; they have the same absolute value but differ in sign. If $m = 0$, both are zero; if $m < 0$, they are imaginary.

93. Solving a Quadratic Equation

To find the solution of the reduced quadratic

$$x^2+px+q=0$$

it suffices to transpose the constant term to the right member of the equation and to add $\left(\frac{p}{2}\right)^2$ to both members of the equation. Then the left member becomes a perfect square and we get the equivalent equation

$$\left(x+\frac{p}{2}\right)^2=\left(\frac{p}{2}\right)^2-q$$

It differs from the simplest equation $x^2=m$ (see Sec. 92) in aspect alone: $x+\frac{p}{2}$ stands in place of x and $\left(\frac{p}{2}\right)^2-q$ in place of m. We find

$$x+\frac{p}{2}=\pm\sqrt{\left(\frac{p}{2}\right)^2-q}$$

Whence

$$\boxed{x=-\frac{p}{2}\pm\sqrt{\left(\frac{p}{2}\right)^2-q}} \qquad (1)$$

This formula shows that every quadratic equation has two roots. These roots may be imaginary if $\left(\frac{p}{2}\right)^2<q$. It may also happen that both roots of the quadratic are the same; this occurs when $\left(\frac{p}{2}\right)^2=q$.

Formula (1) is a particularly convenient form to use when p is an even integer.

Example 1. $x^2-12x-28=0$.

Here $p=-12$, $q=-28$;

$$x=6\pm\sqrt{6^2+28}=6\pm\sqrt{64}=6\pm8,$$
$$x_1=6+8=14,$$
$$x_2=6-8=-2$$

Example 2. $x^2+12x+10=0$.

$$x=-6\pm\sqrt{36-10}=-6\pm\sqrt{26},$$
$$x_1=-6+\sqrt{26}\approx-0.9,\ x_2=-6-\sqrt{26}\approx-11.1$$

Example 3. $x^2-2mx+m^2-n^2=0$.

$x=m\pm\sqrt{m^2-(m^2-n^2)}=m\pm\sqrt{n^2}=m\pm n$,

$x_1=m+n,\ x_2=m-n$.

Note. In Example 2, both roots are negative real numbers, but irrational (Sec. 91). The square roots obtained in solving quadratics may be extracted by means of computations (see Sec. 58) or by using tables. Unfortunately, most problem books give exercises in quadratic equations that are specially constructed so that the roots are extracted exactly. In practical situations, this occurs very rarely. For this reason we strongly advise the student to rid himself of the fear of irrational solutions instilled by such problem books.

When p is not an even integer, it is preferable, when solving reduced quadratics, to use the more general formula (3) given below and assume $a=1$ (see Example 5 below).

The unreduced complete quadratic

$$ax^2+bx+c=0 \qquad (2)$$

may be solved by the formula

$$\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}} \qquad (3)$$

It is obtained from (1) by dividing both members of the unreduced equation (2) by a.

Example 4. $3x^2-7x+4=0$, $(a=3,\ b=-7,\ c=4)$.

$$x=\frac{7\pm\sqrt{7^2-4\cdot 3\cdot 4}}{2\cdot 3}=\frac{7\pm\sqrt{1}}{6},$$

$$x_1=\frac{7+1}{6}=\frac{4}{3},\quad x_2=\frac{7-1}{6}=1$$

Example 5. $x^2+7x+12=0$, $(a=1,\ b=7,\ c=12)$.

$$x=\frac{-7\pm\sqrt{49-4\cdot 12}}{2},$$

$$x_1=-3,\quad x_2=-4$$

Example 6. $0.60x^2+3.2x-8.4=0$.

$$x\approx\frac{-3.2\pm\sqrt{(-3.2)^2-4\cdot 0.60\cdot(-8.4)}}{2\cdot 0.60},$$

$$x_1\approx\frac{-3.2+5.5}{2\cdot 0.60}\approx 1.9,\quad x_2\approx\frac{-3.2-5.5}{2\cdot 0.60}\approx -7.2$$

In Example 6, the coefficients are assumed to be approximate numbers, as is evident from the fact that we write $0.60x^2$ (and not $0.6x^2$). Therefore, it is advisable to perform the operations by the short-cut method given in Secs. 47 and 48. At any rate, note particularly that by the rules of the indicated sections, only two exact significant digits can be obtained. Note that these results are correct to 0.1, but this does not mean that by putting them in the left member of the equation we get a number equal to zero to within 0.1. On the contrary, substituting into the left member, say, the value $x = 1.9$, we get

$$0.60 \cdot 1.9^2 + 3.2 \cdot 1.9 - 8.4 \approx -0.2$$

But if the value of x is increased by 0.1 and we take $x = 2.0$, then we have

$$0.60 \cdot 2.0^2 + 3.2 \cdot 2.0 - 8.4 \approx 0.4$$

Thus, for $x = 1.9$ the left member was negative; for $x = 2.0$ it is positive. This means that it is equal to zero for some value of x lying between 1.9 and 2.0. Consequently, taking $x = 1.9$ we err by not more than 0.1. This is what we mean when we say that the root is equal to 1.9 to within an accuracy of 0.1.

If b is an even number, it is best to give the general formula in the form

$$\boxed{x = \frac{-\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - ac}}{a}}$$

Example 7. $3x^2 - 14x - 80 = 0$.

$$x = \frac{7 \pm \sqrt{7^2 + 3 \cdot 80}}{3} = \frac{7 \pm \sqrt{289}}{3} = \frac{7 \pm 17}{3},$$

$$x_1 = 8, \quad x_2 = -\frac{10}{3}$$

This formula is also convenient when the coefficients a, b, c, are literal expressions.

Example 8. $ax^2 - 2(a+b)x + 4b = 0$.

$$x = \frac{a+b \pm \sqrt{(a+b)^2 - 4ab}}{a} = \frac{a+b \pm \sqrt{a^2 - 2ab + b^2}}{a} = \frac{a+b \pm (a-b)}{a},$$

$$x_1 = 2, \quad x_2 = 2\frac{b}{a}$$

94. Properties of the Roots of a Quadratic Equation

The formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

indicates that in the solution of the quadratic equation $ax^2+bx+c=0$ three cases are possible:

(1) $b^2-4ac>0$; two roots are real and distinct;

(2) $b^2-4ac=0$; two roots are real and equal $\left(\text{both are equal to } -\frac{b}{2a}\right)$;

(3) $b^2-4ac<0$; both roots are imaginary.

The expression b^2-4ac which permits us to discriminate between the three cases is termed the *discriminant*.

The signs of the roots, when the roots are real (that is, when $b^2-4ac\geqslant 0$) are best judged on the basis of the following rule of roots.

The sum of the roots of the reduced quadratic equation

$$x^2+px+q=0$$

is equal to the coefficient of the unknown to the first power with sign reversed, that is,

$$x_1+x_2=-p$$

The product of the roots is equal to the constant term, i.e.,

$$x_1x_2=q$$

95. Factoring a Trinomial of the Type ax^2+bx+c

The quadratic trinomial ax^2+bx+c may be decomposed into first-degree factors in the following manner: solve the quadratic equation $ax^2+bx+c=0$. If x_1 and x_2 are roots of this equation, then $ax^2+bx+c=a(x-x_1)(x-x_2)$.

Example 1. Factor the trinomial $2x^2+13x-24$ into first-degree factors. Solve the equation $2x^2+13x-24=0$. We find the roots $x_1=\frac{3}{2}$, $x_2=-8$. Consequently

$$2x^2+13x-24=2\left(x-\frac{3}{2}\right)(x+8)=(2x-3)(x+8)$$

Example 2. Factor x^2+a^2; the equation $x^2+a^2=0$ has imaginary roots: $x_1=\sqrt{-a^2}$, $x_2=-\sqrt{-a^2}$ and so it is impossible to factor x^2+a^2 into real factors of the first degree. It can be factored into imaginary factors, however:

$$x^2+a^2=(x+\sqrt{-a^2})(x-\sqrt{-a^2})=(x+ai)(x-ai)$$

(i denotes the imaginary number $\sqrt{-1}$)

96. Higher-Degree Equations Solvable by Means of Quadratics

Some algebraic equations of higher degree can be solved by reducing them to quadratic equations. The most important cases are the following.

1. The left member of an equation can sometimes readily be decomposed into factors one of which is a polynomial not higher than the second degree. Then we solve the resulting equations by equating each factor to zero separately. The roots thus found will be the roots of the original equation.

Example 1. $x^4+5x^3+6x^2=0$.

The polynomial $x^4+5x^3+6x^2$ can readily be factored into x^2 and (x^2+5x+6). We solve the equation $x^2=0$ to get two equal roots $x_1=x_2=0$. Solve the equation $x^2+5x+6=0$. Denoting its roots by x_3 and x_4, we have $x_3=-2$, $x_4=-3$. The roots of the original equation are $x_1=x_2=0$, $x_3=-2$, $x_4=-3$.

Example 2. Solve the equation $x^3=8$.

Rewriting it as $x^3-8=0$, we factor the left member: $x^3-8=(x-2)(x^2+2x+4)$. The equation $x-2=0$ yields $x_1=2$, the equation $x^2+2x+4=0$ yields $x_2=-1+\sqrt{-3}$, $x_3=-1-\sqrt{-3}$. Thus, the equation $x^3=8$ has one real root and two imaginary roots. In other words $\sqrt[3]{8}$ has, besides the obvious real value of 2, two imaginary roots (see Sec. 111, Example 3).

2. If the equation is of the form $ax^{2n}+bx^n+c=0$, it can be reduced to a quadratic by introducing a new unknown thus: $x^n=z$.

Example 3. $x^4-13x^2+36=0$. Rewriting the equation as $(x^2)^2-13x^2+36=0$ we introduce the new unknown $x^2=z$. The equation then becomes $z^2-13z+36=0$. Its roots are $z_1=9$, $z_2=4$. Now solve the equations $x^2=9$ and $x^2=4$. The first has the roots $x_1=3$, $x_2=-3$, the second, the roots $x_3=2$, $x_4=-2$. The roots of the given equation are: 3, —3, 2, —2.

It is thus possible to solve any equation of the form $ax^4+bx^2+c=0$, which is called a *biquadratic equation.*

Example 4. $x^6-16x^3+64=0$. Representing the equation as $(x^3)^2+16x^3+64=0$, introduce a new unknown: $x^3=z$. This yields the equation $z^2-16z+64=0$, which has two equal roots $z_1=z_2=8$. Solve $x^3=8$ to get (see Example 2) $x_1=2$, $x_2=-1+\sqrt{-3}$, $x_3=-1-\sqrt{-3}$. The other three roots in this case are (since $z_1=z_2$) equal to these three.

97. A System of Quadratic Equations in Two Unknowns

The most general form of a second-degree equation in two unknowns is

$$ax^2+bxy+cy^2+dx+ey+f=0$$

where a, b, c, d, e, f are given numbers or expressions involving known quantities. One equation of the second degree in two unknowns has an infinity of solutions (see Sec. 85).

A system of two equations in two unknowns, one equation being a quadratic equation and the other a first-degree equation, may be solved by the substitution method described in Sec. 86. The expression of one unknown in terms of the other is found from the first-degree equation. Substituting this expression into the second-degree equation, we get an equation in one unknown. In the general case, it is a quadratic equation (see Example 1). However, it may happen that the second-degree terms cancel out; we then have a first-degree equation (see Example 2).

Example 1. $x^2-3xy+4y^2-6x+2y=0$, $x-2y=3$.

From the second equation we find $x=3+2y$. Substituting this expression into the first equation, we get

$$(3+2y)^2-3(3+2y)y+4y^2-6(3+2y)+2y=0$$

Solving this equation, we find

$$9+12y+4y^2-9y-6y^2+4y^2-18-12y+2y=0,$$

$$2y^2-7y-9=0,$$

$$y=\frac{7\pm\sqrt{49+72}}{4},$$

$$y_1=\frac{9}{2},\quad y_2=-1$$

Insert the values thus found: $y_1=\frac{9}{2}$, $y_2=-1$ into the expression $x=3+2y$ to get $x_1=12$, $x_2=1$.

Example 2. $x^2-y^2=1$, $x+y=2$.

From the second equation we find $y=2-x$. Putting this expression into the first equation, we get $x^2-(2-x)^2=1$. After collecting terms, second-degree terms cancel out, and we have $-4+4x=1$, whence $x=\frac{5}{4}$. Putting this value into the expression $y=2-x$, we find $y=\frac{3}{4}$.

A system of two quadratic equations in two unknowns can be solved thus: if one of the equations does not contain the term ax^2 (or the term cy^2), then use the substitution method expressing x (or y) of this equation in terms of y (or x); but if both equations involve terms like ax^2 and cy^2, then first apply the method of addition or subtraction (Sec. 85) so as to obtain an equation without ax^2 or cy^2. Then apply the substitution method. After the elimination process we have an equation in one unknown (generally speaking, of the fourth degree). Only in exceptional cases does it reduce to a quadratic equation, but these cases are encountered rather frequently in the solution of geometric problems.

Example 3.

$$x^2+xy+2y^2=74, \quad 2x^2+2xy+y^2=73$$

Both equations have terms involving x^2 and terms involving y^2. So first use the addition-or-subtraction method to obtain an equation without, say, y^2.

$$\begin{array}{r|c|cl} 2x^2+2xy+y^2=73 & 2 & - & 4x^2+4xy+2y^2=146 \\ x^2+xy+2y^2=74 & & & x^2 \mp xy \mp 2y^2=-74 \\ \cline{2-4} & & & 3x^2+3xy \qquad\quad =72 \end{array}$$

From this equation we find the expression of y in terms of x

$$y=\frac{24-x^2}{x}$$

Then we substitute this expression into one of the given equations, say the first, to get

$$x^2+x\,\frac{24-x^2}{x}+2\,\frac{(24-x^2)^2}{x^2}=74$$

Simplifications yield

$$x^4+24x^2-x^4+1152-96x^2+2x^4=74x^2,$$
$$2x^4-146x^2+1152=0,$$
$$x^4-73x^2+576=0$$

We have a biquadratic equation (see Sec. 96, Example 3). Putting $x^2=z$, we reduce it to the equation $z^2-73z+576=0$ which yields

$$z=\frac{73\pm\sqrt{73^2-4\cdot576}}{2}=\frac{73\pm\sqrt{3025}}{2}=\frac{73\pm55}{2},$$
$$z_1=64,\quad z_2=9$$

The first solution gives us $x_1=8$, $x_2=-8$, the second, $x_3=3$, $x_4=-3$. Putting the values x_1, x_2, x_3, x_4 into the expression $y=\frac{24-x^2}{x}$, we get the corresponding values of y:

$$y_1=-5,\quad y_2=+5,\quad y_3=+5,\quad y_4=-5$$

In solving systems of second-degree equations artificial techniques often yield faster and more elegant results.

98. Complex Numbers

The development of algebra (Sec. 66) called for numbers of a new kind besides the familiar positive and negative numbers. They received the name *complex numbers*.

A complex number has the form $a+bi$, where a and b are real numbers and i is a new kind of number called the *imaginary unit*. "Imaginary" numbers (see Sec. 92) constitute a special subset of the complex numbers (when $a=0$). On the other hand, the real numbers (positive and negative) also constitute a special subset of the complex numbers (when $b=0$).

Let us call the real number a the *abscissa* of the complex number $a+bi$; the real number b, the *ordinate* of the complex number $a+bi$. The basic property of the number i is that the product $i\cdot i$ is equal to -1, that is

$$i^2=-1 \tag{1}$$

For a long time, no physical quantities could be found that obeyed the same rules as those involving complex numbers, say Rule (1). Whence the name "imaginary unit", "imaginary number", etc. At present we know many such physical quantities, and complex numbers are extensively employed not only in mathematics but also in physics and engineering (theory of elasticity, electrical engineering, aerodynamics, etc.).

Below (Sec. 104) a geometric interpretation of complex numbers is given, but first the rules for operating with them are considered (Secs. 100-103). Here, the question of the geometric or physical meaning of the number i is not considered because it differs in various spheres of science.

The rule for each operation involving complex numbers is derived from the definition of the operation. The definitions of the operations involving complex numbers were not devised arbitrarily but were established to ensure consistency with the rules for operating with real numbers (cf. Sec. 34). The point is that complex numbers must not be considered separately from the real numbers.

99. Basic Conventions Concerning Complex Numbers

1. The real number a can also be written as $a+0\cdot i$ (or $a-0\cdot i$).

Examples. The notation $3+0\cdot i$ means the same as 3. The notation $-2+0\cdot i$ means -2. The notation $\frac{3\sqrt{2}}{2}+0\cdot i$ is equivalent to $\frac{3\sqrt{2}}{2}$.

Note. We have similar instances in ordinary arithmetic where the fraction $\frac{5}{1}$ denotes the same thing as 5. The notation 002 is simply 2, etc.

2. A complex number of the form $0+bi$ is called a pure imaginary. The notation bi is the same as $0+bi$.

3. Two complex numbers $a+bi$, $a'+b'i$ are considered equal if they have equal abscissas and ordinates, that is, if $a=a'$, $b=b'$. Otherwise these complex numbers are not equal. This definition is suggested by the following reasoning. If, say, we could have an equation like $2+5i=8+2i$, then by the rules of algebra we would get $i=2$, whereas i cannot be a real number.

Note. We have not yet defined the addition of complex numbers, and so, strictly speaking, we cannot yet assert that the number $2+5i$ is the sum of the numbers 2 and $5i$. It would be more correct to say that we have here a *pair of real numbers* 2 (abscissa) and 5 (ordinate); these numbers generate a new kind of number which we agree to denote by $2+5i$.

100. Addition of Complex Numbers

Definition. The sum of the complex numbers $a+bi$ and $a'+b'i$ is the complex number $(a+a')+(b+b')i$.

This definition is suggested by the rules for operating on ordinary polynomials.

Example 1. $(-3+5i)+(4-8i)=1-3i$.

Example 2. $(2+0i)+(7+0i)=9+0i$. Since (Sec. 99) the notation $2+0i$ means the same as 2, etc., the operation performed is in agreement with ordinary arithmetic $(2+7=9)$.

Example 3. $(0+2i)+(0+5i)=0+7i$, that is to say, (Sec. 99), $2i+5i=7i$.

Example 4. $(-2+3i)+(-2-3i)=-4$.

In Example 4 the sum of two complex numbers is equal to a real number. Two complex numbers like $a+bi$ and $a-bi$ are called *conjugate complex numbers*. The sum of the two conjugate complex numbers is equal to the real number $2a$. Note that the sum of two nonconjugate complex numbers can also be a real number, as witness $(3+5i)+(4-5i)=7$.

Note. Now that the operation of addition has been defined, we can consider the complex number $a+bi$ as the sum of the numbers a and bi. Thus, the number 2 (which we agreed could be written as $2+0i$) and the number $5i$ (which, by Sec. 99, means the same as $0+5i$) yield a sum equal (according to the definition) to $2+5i$.

101. Subtraction of Complex Numbers

Definition. The difference between the complex number $a+bi$ (minuend) and $a'+b'i$ (subtrahend) is the complex number $(a-a')+(b-b')i$.

Example 1. $(-5+2i)-(3-5i)=-8+7i$.

Example 2. $(3+2i)-(-3+2i)=6+0i=6$.

Example 3. $(3-4i)-(3+4i)=-8i$.

Note. The subtraction of complex numbers can be defined as an operation inverse to addition. Namely, we seek a complex number $x+yi$ (difference) such that $(x+yi)+(a'+b'i) = a+bi$. By the definition of Sec. 100, we have

$$(x+a')+(y+b')\,i=a+bi$$

By the condition for the equality of complex numbers (Sec. 99),

$$x+a'=a, \quad y+b'=b$$

From these equations we find $x=a-a'$, $y=b-b'$

102. Multiplication of Complex Numbers

The definition of the multiplication of complex numbers is devised so that (1) the numbers $a+bi$ and $a'+b'i$ can be multiplied as algebraic binomials and (2) the number i has the property $i^2=-1$. By virtue of Requirement 1, the product $(a+bi)(a'+b'i)$ must be equal to $aa'+(ab'+ba')\,i + bb'i^2$, and by virtue of Requirement 2, this expression must equal $(aa'-bb')+(ab'+ba')\,i$. Accordingly we have the following definition.

Definition. The product of the complex numbers $a+bi$ and $a'+b'i$ is the complex number

$$(aa'+bb')+(ab'+ba')\,i \tag{1}$$

Note 1. The equality $i^2=-1$ was in the nature of a requirement prior to establishment of the rules for multiplication. Now however it follows from the definition. The point is that the notation i^2, that is, $i \cdot i$, is equivalent (Sec. 99) to the notation $(0+1\cdot i)(0+1\cdot i)$. Here, $a=0$, $b=1$, $a'=0$, $b'=1$. We have $aa'-bb'=-1$, $ab'+ba'=0$, so that the product is $-1+0i$, that is, -1.

Note 2. In practical situations there is no need to use formula (1). One can multiply the given numbers as binomials and then put $i^2=-1$.

Example 1. $(1-2i)(3+2i) = 3-6i+2i-4i^2=3-6i + 2i+4=7-4i$.

Example 2. $(a+bi)(a-bi)=a^2+b^2$.

Example 2 shows that the product of conjugate complex numbers is a real positive number. *

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* But the product of two nonconjugate complex numbers can also be a positive real number; for example $(2+3i)(4-6i)=26$ (see Sec. 100). Now if both the sum and the product of two complex numbers are real numbers, then these complex numbers must be conjugate.

103. Division of Complex Numbers

In accordance with the definition of the division of real numbers we give the following definition.

Definition. To divide a complex number $a+bi$ (dividend) by a complex number $a'+b'i$ (divisor) means to find a number $x+yi$ (quotient) such that, when multiplied by the divisor, it yields the dividend.

If the divisor is nonzero, then division is always possible and the quotient is unique (for proof see Note 2). A convenient practical way to find the quotient is this.

Example 1. Find the quotient of $(7-4i):(3+2i)$.

Writing the fraction $\frac{7-4i}{3+2i}$, multiply both terms by number $3-2i$, which is the conjugate of $3+2i$ (see Sec. 102, Example 1), to get

$$\frac{(7-4i)(3-2i)}{(3+2i)(3-2i)}=\frac{13-26i}{13}=1-2i$$

Example 1 of the preceding section is a check.

Example 2. $\frac{-2+5i}{-3-4i}=\frac{(-2+5i)(-3+4i)}{(-3-4i)(-3+4i)}=\frac{-14-23i}{25}=-0.56-0.92i$.

Example 3. $\frac{-6+21i}{4-14i}=-\frac{3}{2}$. The easiest way here is to divide both terms by $(-2+7i)$.

Doing as we did in Examples 1 and 2, we find the general formula

$$(a+bi):(a'+b'i)=\frac{aa'+bb'}{a'^2+b'^2}+\frac{a'b+b'a}{a'^2+b'^2}i \tag{1}$$

In order to prove that the right member of (1) is really the quotient, multiply it by $a'+b'i$ to get $a+bi$.

Note 1. We could take formula (1) as the definition of division (cf. the definitions in Secs. 100 and 101).

Note 2. Formula (1) can also be derived as follows. By the definition we must have $(a'+b'i)(x+yi)=a+bi$. Hence (Sec. 99) the following two equations must be satisfied:

$$a'x-b'y=a, \quad b'x+a'y=b \tag{2}$$

This system has a unique solution:

$$x=\frac{aa'+bb'}{a'^2+b'^2}, \quad y=\frac{a'b+b'a}{a'^2+b'^2}$$

if $\frac{a'}{b'}\neq-\frac{b'}{a'}$ (Sec. 87), that is, if $a'^2+b'^2\neq 0$.

It remains to consider the case $a'^2+b'^2=0$. This case (the numbers a' and b' are real!) is only possible when $a'=0$ and $b'=0$, that is when the divisor $a'+b'i$ is zero. If however the dividend $a+bi$ is also zero, then the quotient is indeterminate (see Sec. 37, Item 2.) But if the dividend is not zero, then the quotient does not exist (we say it is equal to infinity) (cf. Sec. 37, Item 3).

104. Complex Numbers Depicted in the Plane

Real numbers may be depicted by points on a straight line, as shown in Fig. 2 where point A denotes number 4 and point B number -5. These same numbers can also be indicated by the line-segments OA, OB, which indicate both length and direction.

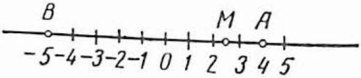

Fig. 2.

Every point M of the number line depicts some real number (rational if the line segment OM is commensurable with the unit length, and irrational if it is incommensurable). Thus, there is no place for complex numbers on the number line.

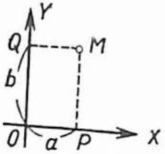

Fig. 3

But complex numbers can be depicted on a "number plane". To do this we choose a rectangular system of coordinates in the plane (see Sec. 211) with the same scale on both axes (Fig. 3). The complex number $a+bi$ is depicted by a point M, whose abscissa x (in Fig. 3, $x=OP=QM$) is equal to the abscissa a of the complex number and the ordinate $y\,(OQ=PM)$ is equal to the ordinate b of the complex number.

Examples. In Fig. 4, point A with abscissa $x=3$ and ordinate $y=5$ depicts the complex number $3+5i$. Point B depicts the complex number $-2+6i$; point C is the complex number $-6-2i$; point D denotes the complex number $2-6i$.

Real numbers (in complex form they look like this: $a+0i$) are depicted by points on the X-axis and pure imaginary numbers (of the form $0+bi$), by points on the Y-axis.

Examples. Point K in Fig. 4 depicts the real number 6 (or, what is the same thing, the number $6+0i$), point L depicts the pure imaginary number $3i$ (i. e., $0+3i$); point N, the pure imaginary $-4i$ (i. e., $0-4i$). The origin of the coordinates indicates the number 0 (which is $0+0i$).

Conjugate complex numbers are depicted by a pair of points symmetric about the axis of abscissas (X-axis); thus, the points C and C' in Fig. 4 indicate the conjugate numbers $-6-2i$ and $-6+2i$.

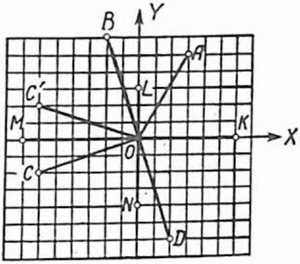

Fig. 4.

Complex numbers can also be indicated by line-segments (vectors) issuing from the point O and terminating at the appropriate point of the number plane. Thus, the complex number $-2+6i$ may be described not only by the point B (Fig. 4), but also by the vector OB. The complex number $-6-2i$ is denoted by the vector OC, etc.

Note. When we call some line-segment a *vector* we mean that there are two essential things about the line: it has length and it has a certain *direction*. Two vectors are considered the same (equal) only when they have the same length and the same direction.

105. The Modulus and Argument of a Complex Number

The length of the vector depicting a complex number is called the *modulus* of the complex number. The modulus of any complex number not equal to zero is a positive number. The modulus of the complex number $a+bi$ is symbolized as $|a+bi|$ and also by the letter r. From the drawing (Fig. 5) it is clear that

$$r=|a+bi|=\sqrt{a^2+b^2} \qquad (1)$$

The modulus of a real number coincides with its absolute value. The conjugate complex numbers $a+bi$ and $a-bi$ have the same modulus.

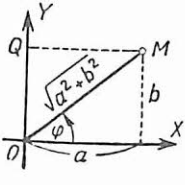

Fig. 5.

Example 1. The modulus of the complex number $3+5i$ (that is, the length of the vector OA in Fig. 4) is equal to $\sqrt{3^2+5^2}=\sqrt{34}\approx 5.83$.

Example 2. $|1+i|=\sqrt{1^2+1^2}=\sqrt{2}\approx 1.41$.

Example 3. $|-3+4i|=5$.

Example 4. The modulus of the number -7 (that is, $-7+0i$) is the length of the vector OM (Fig.4). This length is expressed by the positive number 7, or

$$|-7+0i|=\sqrt{(-7)^2+0^2}=7$$

Example 5. The modulus of the number $-4i$ (length of vector ON, Fig. 4) is 4.

Example 6. The modulus of the number $-6-2i$ (length of vector OC, Fig. 4) is equal to $\sqrt{40}\approx 6.32$. The modulus of the number $-6+2i$ (length of vector OC', Fig. 4) is also equal to $\sqrt{40}$. Two conjugate complex numbers always have equal moduli.

The angle φ between the axis of abscissas and the vector OM describing the complex number $a+bi$ is called the *argument* of the complex number $a+bi$. In Fig. 6, the vector OM depicts the complex number $-3-3i$. The angle XOM is the argument of this complex number.

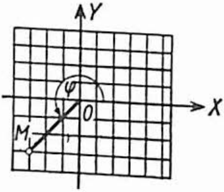

Fig. 6.

Every nonzero complex number (for the number 0 the argument is indeterminate) has an infinite number of arguments that differ by an integral number of complete rotations (that is, by 360°k, where k is any integer). Thus, the arguments of the complex numbers $-3-3i$ are all angles of the form $225°\pm 360°k$, for instance, $225°+360°=585°$, $225°-360°=-135°$.

The argument φ is connected with the coordinates of a complex number $a+bi$ by the following formulas (see Fig. 5):

$$(1)\ \tan\varphi=\frac{b}{a},\quad (2)\ \cos\varphi=\frac{a}{\sqrt{a^2+b^2}}, \qquad (4)$$

$$(3)\ \sin\varphi=\frac{b}{\sqrt{a^2+b^2}}$$

However, none of these formulas, taken separately, permits finding the argument from the abscissa and ordinate (see examples).

Example 7. Find the argument of the complex number $-3-3i$.

By formula (2) $\tan\varphi=\frac{-3}{-3}=1$. This condition is satisfied by an angle of 45° and also by an angle of 225°. But the 45° angle is not the argument of the number $-3-3i$ (Fig. 6). The correct answer is $\varphi=225^\circ$ (or -135°, or 585°, etc.). This result is obtained if we note that the abscissa and the ordinate of the given complex number are negative. This means that point M lies in the third quadrant.

Alternative method. Use formula (3) to find $\cos\varphi=\frac{-1}{\sqrt{2}}$. Formula (4) shows that $\sin\varphi$ is also negative. Hence the angle φ belongs to the third quadrant so that $\varphi=225^\circ\pm360^\circ k$.

Example 8. Find the argument of the complex number $-2+6i$. We find $\tan\varphi=\frac{6}{-2}=-3$. Since the abscissa is negative and the ordinate is positive, the angle φ lies in the second quadrant. Using tables we find $\varphi\approx180^\circ-72^\circ=108^\circ$. See Fig. 4 where point B depicts $-2+6i$.

The least (in absolute value) argument is called the *principal value* of the argument. Thus, for the complex numbers $-3-3i$, $2i-5i$, the principal values of the argument are -135°, $+90^\circ$, -90°.

The argument of a positive real number has the principal value 0°; for negative numbers, the principal value of the argument is taken to be 180° (and not -180°).

In the case of conjugate complex numbers, the principal values of the argument have the same absolute values but opposite signs. Thus, the principal values of the argument of the numbers $-3+3i$ and $-3-3i$ are 135° and -135°.

106. Trigonometric Form of a Complex Number

Abscissa a and ordinate b of the complex number $a+bi$ are expressed in terms of the modulus r and the argument φ (see Fig. 5) by the formulas

$$a=r\cos\varphi,\quad b=r\sin\varphi$$

Therefore, any complex number can be represented in the form $r(\cos\varphi+i\sin\varphi)$ where $r\geqslant0$.

This expression is termed the *normal trigonometric form* or, simply, the trigonometric form of a complex number.

Example 1. Represent the complex number $-3-3i$ in the normal trigonometric form. We have (Sec. 105)

$$r=\sqrt{(-3)^2+(-3)^2}=3\sqrt{2}$$

Consequently

$$-3-3i=3\sqrt{3}\,(\cos(-135^\circ)+i\sin(-135^\circ))$$

or

$$-3-3i=3\sqrt{2}\,(\cos 225^\circ+i\sin 225^\circ)$$

and so forth.

Example 2. For the complex number $-2+6i$ we have

$$r=\sqrt{(-2)^2+6^2}=\sqrt{40}$$

and (Sec. 105, Example 2) $\varphi=108^\circ$. Hence, the normal trigonometric form of the number $-2+6i$ is

$$\sqrt{40}\,(\cos 108^\circ+i\sin 108^\circ)$$

Example 3. The normal trigonometric form of the number 3 is $3(\cos 0^\circ+i\sin 0^\circ)$ or, in general form,

$$3(\cos 360^\circ k+i\sin 260^\circ k)$$

Example 4. The normal trigonometric form of the number -3 is $3(\cos 180^\circ+i\sin 180^\circ)$ or

$$3[\cos(180^\circ+360^\circ k)+i\sin(180^\circ+360^\circ k)]$$

Example 5. The normal trigonometric form of the imaginary unit i is $\cos 90^\circ+i\sin 90^\circ$ or

$$\cos(90^\circ+360^\circ k)+i\sin(90^\circ+360^\circ k)$$

Here $r=1$.

Example 6. The normal trigonometric form of the number $-i$ is $\cos(-90^\circ)+i\sin(-90^\circ)$ or

$$\cos(-90^\circ+360^\circ k)+i\sin(-90^\circ+360^\circ k)$$

Here $r=1$.

In contrast to the trigonometric form, an expression like $a+bi$ is called the *algebraic* or *coordinate* (Cartesian) form of the complex number.

Example 7. Represent the complex number $2[\cos(-40^\circ)+i\sin(-40^\circ)]$ in algebraic form.

Here $r=2$, $\varphi=-40^\circ$. By formulas (3) and (4) of the preceding section,

$$a=r\cos\varphi=2\cos(-40^\circ)\approx 2\cdot 0.766=1.532,$$
$$b=r\sin\varphi=2\sin(-40^\circ)\approx 2\cdot(-0.643)=-1.286$$

The algebraic form of the given number is (approximately) $1.532-1.286i$.

Example 8. Represent in algebraic form the number $3(\cos 270^\circ + i\sin 270^\circ)$. Since $\cos 270^\circ = 0$, $\sin 270^\circ = -1$, the given number is equal to $-3i$.

Example 9. If $r(\cos\varphi + i\sin\varphi)$ is one of two conjugate complex numbers, then the other can be represented in the form $r[\cos(-\varphi) + i\sin(-\varphi)]$ or in the form $r(\cos\varphi - i\sin\varphi)$; incidentally, the latter expression is not the normal form.

107. Geometric Meaning of Addition and Subtraction of Complex Numbers

Let the vectors OM and OM' (Fig. 7) depict the complex numbers $z = x + yi$ and $z' = x' + y'i$. From the point M draw the vector MK equal to OM' (that is, having the same length and the same direction as OM'; see Sec. 104, note). Then the vector OK gives the sum of the given complex numbers.[*]

The vector OK thus constructed is termed the *geometric sum* (or, briefly, the sum) of the vectors OM and OM' (the name "sum" is due to the fact that it arises by analogy with the combining of velocities of moving bodies, of forces applied to a point, and of many other physical quantities).

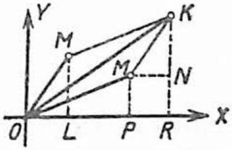

Fig. 7.

Thus, the sum of two complex numbers is given by the sum of the vectors depicting the separate summands.

The length of side OK of the triangle OMK is less than the sum and greater than the difference of the lengths OM and MK. Therefore

$$\big||z|-|z'|\big| \leqslant |z+z'| \leqslant |z|+|z'|$$

The equality is valid only when the vectors OM and OM' have the same (Fig. 8) or opposite directions (Fig. 9). In the former case, $|OM|+|OM'|=|OK|$, that is $|z+z'| = |z|+|z'|$. In the latter case $|z+z'| = \big||z|-|z'|\big|$.

---

[*] Indeed, the triangles $OM'L$ and MKN are equal. Hence, $x' = OL = MN = PR$, $y' = LM' = NK$. Consequently, the abscissa $OR = OP + PR = x + x'$, and the ordinate $RK = y + y'$.

Example 1. Let $z=4+3i$, $z'=5+12i$. Then

$$|z|=\sqrt{4^2+3^2}=5,\ |z'|=\sqrt{5^2+12^2}=13,\quad z+z'=9+15i,$$

$$|z+z'|=\sqrt{9^2+15^2}=\sqrt{306}$$

We have $13-5<\sqrt{306}<13+5$, $8<\sqrt{306}<18$.

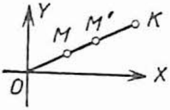

Fig. 8.

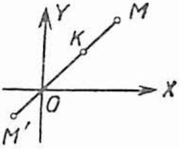

Fig. 9.

Example 2. Let $z=4+3i$, $z'=8+6i$.
These complex numbers have the same argument (36°52'), the corresponding vectors are in the same direction.
Here

$$|z|=5,\ |z'|=10,\ z+z'=12+9i,$$

$$|z+z'|=\sqrt{12^2+9^2}=15$$

We have $10-5<15=10+5$.

Example 3. Let $z=8-6i$, $z'=-12+9i$.
These complex numbers are depicted by vectors of opposite direction (their arguments are equal to 323°08' and 143°08').
Here

$$|z|=10,\ |z'|=15,\ z+z'=-4+3i,\ |z+z'|=5$$

We have

$$15-10=5<15+10$$

The *sum of three or more complex numbers* can also be represented as the sum of vectors (OM, OM', OM'' in Fig. 10) depicting the separate summands, that is to say, the vector OK which completes the polygonal line $OMSK$ (the vector MS is equal to the vector OM'; the vector SK equals the vector OM''). The summands may be taken in any order; the polygonal lines will be different but their extremities will coincide. Since OK is not longer than the polygonal line $OMSK$, it follows that

$$|z+z'+z''|\leqslant|z|+|z'|+|z''|$$

The equality is valid only when all summands have the same direction.

The difference between the complex numbers $a+bi$ and $a'+b'i$ is equal to the sum of the numbers $a+bi$ and $-a'-b'i$. The latter summand has the same modulus as $a'+b'i$ but is opposite in direction. Therefore, the difference between two complex numbers represented by the vectors OM and OM' (Fig. 11) is depicted as the sum of the vectors OM and OM'' (as the vector OT).

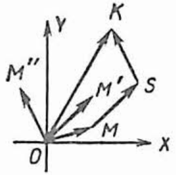

Fig. 10

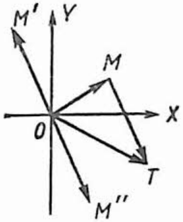

Fig. 11

108. Geometric Meaning of Multiplication of Complex Numbers

Let two complex numbers z and z' be depicted by the vectors OM and OM' (Fig. 12). Write down the factors in trigonometric form and compute the product:

$$\begin{aligned} zz' &= r(\cos\varphi + i\sin\varphi)\cdot r'(\cos\varphi' + i\sin\varphi') \\ &= rr'[(\cos\varphi\cos\varphi' - \sin\varphi\sin\varphi') + i(\sin\varphi\cos\varphi' \\ &\quad + \cos\varphi\sin\varphi')] \end{aligned}$$

That is (Sec. 194)

$$zz' = rr'[\cos(\varphi+\varphi') + i\sin(\varphi+\varphi')] \tag{1}$$

The modulus of the product (it is depicted by the vector OL) is rr', and the argument of the product is equal to $\varphi+\varphi'$. In other words, *to multiply two complex numbers, multiply their moduli and add their arguments.*

This rule holds true for any number of factors.

Example 1. Take the complex numbers depicted by the vectors OM and OM' in Fig. 12. Their moduli are $|OM| = \frac{3}{2}$ and $|OM'| = 2$, and the arguments are $\angle XOM = 20°$ and $\angle XOM' = 30°$. The modulus of the product depicted by the vector OL is $\frac{3}{2}\cdot 2 = 3$; the argument of the product (angle XOL) is $20° + 30° = 50°$. We get

$$\frac{3}{2}(\cos 20° + i \sin 20°)\cdot 2(\cos 30° + i \sin 30°)$$
$$= 3(\cos 50° + i \sin 50°)$$

Example 2.

$$4\sqrt{2}(\cos 45° + i \sin 45°)\cdot\frac{\sqrt{2}}{2}(\cos 135° + i \sin 135°)$$
$$= 4(\cos 180° + i \sin 180°) = -4 \text{ (Fig. 13).}$$

The same factors in algebraic form are $4 + 4i$ and $-\frac{1}{2} + \frac{1}{2}i$. Multiplying them together, we again get -4.

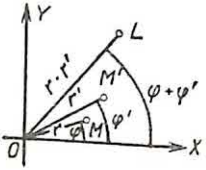

Fig. 12

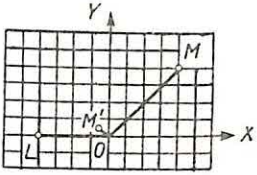

Fig. 13

Example 3. Multiply together $2(\cos 150° + i \sin 150°)$, $3[\cos(-160°) + i \sin(-160°)]$ and $0.5(\cos 10° + i \sin 10°)$. The modulus of the product $2\cdot 3\cdot 0.5 = 3$. The argument of the product $150° - 160° + 10° = 0°$. The product is

$$3(\cos 0° + i \sin 0°) = 3$$

Example 4. $r(\cos\varphi + i\sin\varphi)\cdot r[\cos(-\varphi) + i\sin(-\varphi)] = r^2(\cos 0° + i \sin 0°) = r^2$.
The product of two conjugate complex numbers is a real number equal to the square of their common modulus.

Example 5. $\frac{3}{2}[\cos(-20°) + i \sin(-20°)]\cdot 2[\cos(-30°) + i \sin(-30°)] = 3[\cos(-50°) + i \sin(-50°)]$.
Comparing this with Example 1, we see that by replacing the factors with conjugate numbers the product is replaced with its conjugate number. This property is general and can be extended to any number of factors.

Note 1. The rules for multiplying real numbers are a special case of the above rule. Thus, in multiplying the numbers -2 and -3 their arguments (180° and 180°) combine to form 360° so that the product is the positive number 6 [that is, $6(\cos 360° + i\sin 360°)$].

Note 2. When a complex number $r(\cos\varphi + i\sin\varphi)$ is multiplied by the imaginary unit i (the modulus of which is 1 and the argument $+90°$), the modulus of the product remains equal to r. The argument however is increased by 90°, which means the vector of the factor is rotated through $+90°$ without changing its length. In particular, multiplication of 1 (the vector OA in Fig. 14) by i is represented by a rotation of the vector OA through 90° to position OB, while multiplication of i by i is represented by a rotation of OB through 90° to the position OC. But the vector OC is depicted as -1. Therefore, $i^2 = -1$. In geometric representation, the number i is no more "imaginary" than the number -1.

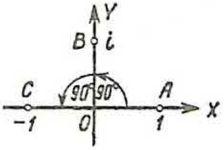

Fig. 14.

109. Geometric Meaning of Division of Complex numbers

Division is the inverse of multiplication. Therefore, (see Sec. 108), when dividing complex numbers, divide their moduli (the modulus of the dividend by the modulus of the divisor) and subtract the arguments (the argument of the divisor from the argument of the dividend), or

$$r(\cos\varphi + i\sin\varphi) : r'(\cos\varphi' + i\sin\varphi') = \frac{r}{r'}[\cos(\varphi - \varphi') + i\sin(\varphi - \varphi')] \qquad (1)$$

Example 1. $2(\cos 30° + i\sin 30°) : 6(\cos 45° + i\sin 45°) = \frac{1}{3}[\cos(-15°) + i\sin(-15°)]$.

Example 2. $-4 : 4\sqrt{2}(\cos 45° + i\sin 45°) = 4(\cos 180° + i\sin 180°) : 4\sqrt{2}(\cos 45° + i\sin 45°) = \frac{1}{\sqrt{2}}(\cos 135° + i\sin 135°)$. Cf. Example 2 of the preceding section.

In algebraic form

$$-4 : (4 + 4i) = \frac{-1}{1+i} = \frac{-1(1-i)}{(1+i)(1-i)} = \frac{-1+i}{2}$$

Example 3. Divide 1 by the complex number $r(\cos\varphi + i\sin\varphi)$. The dividend can be written as $1(\cos 0° + i\sin 0°)$.

By formula (1) the quotient will be $\frac{1}{r}[\cos(-\varphi)+i\sin(-\varphi)]$.

$$1:r(\cos\varphi+i\sin\varphi)=\frac{1}{r}[\cos(-\varphi)+i\sin(-\varphi)] \qquad (2)$$

Geometric construction: describe a circle of radius 1 with centre at O. Let $|r|>1$, that is, the point M (Fig. 15) depicting the divisor lies outside the circle. Draw the tangent MT; from point T draw the perpendicular TM' to OM. The point L which is symmetric with M' about the axis of abscissas depicts the quotient. Indeed, $|OL|=|OM'|$ and from the right triangle OTM, in which TM' is the altitude, we find $|OT|^2=|OM|\cdot|OM'|$; that is, $1=r|OM'|$ or $|OM'|=\frac{1}{r}$. The arguments of the vectors OM and OL are obviously equal in magnitude and opposite in sign.

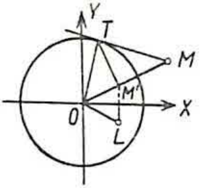

Fig 15.

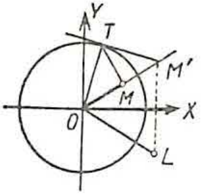

Fig. 16.

For the case $|r|<1$ see construction in Fig. 16.

From formula (2) it follows that division of 1 by a complex number with modulus $r=1$ yields a complex number that is conjugate to the divisor.

Example 4. $2[\cos(-30^\circ)+i\sin(-30^\circ)]:6[\cos(-45^\circ)+i\sin(-45^\circ)]=\frac{1}{3}(\cos 15^\circ+i\sin 15^\circ)$.

Comparing this with Example 1, we see that replacing the dividend and divisor by conjugate numbers replaces the quotient by its conjugate number. Formula (1) shows that this property is general.

110. Raising a Complex Number to an Integral Power

According to Sec. 108,

$$[r(\cos\varphi+i\sin\varphi)]^2=r^2(\cos 2\varphi+i\sin 2\varphi),$$
$$[r(\cos\varphi+i\sin\varphi)]^3=r^3(\cos 3\varphi+i\sin 3\varphi)$$

and generally

$$[r(\cos\varphi+i\sin\varphi)]^n=r^n(\cos n\varphi+i\sin n\varphi) \qquad \text{(A)}$$

where n is a positive integer. Formula (A) is called *De Moivre's theorem* (after Abraham De Moivre, 1667—1754). It is valid for a negative integral exponent n (Sec. 125) and also for $n=0$.

For example, $[r(\cos\varphi+i\sin\varphi)]^{-3}$

$$=\frac{1}{[r(\cos\varphi+i\sin\varphi)]^3}=\frac{1}{r^3(\cos 3\varphi+i\sin 3\varphi)}$$

Consequently (cf. Example 3 of the preceding section),

$$[r(\cos\varphi+i\sin\varphi)]^{-3}=r^{-3}[\cos(-3\varphi)+i\sin(-3\varphi)]$$

To summarize, to raise a complex number to any integral power, raise the modulus to that power and multiply the argument by the exponent of the power. For raising to a fractional power see Sec. 112.

Example 1. Raise to the sixth power the number

$$z=2(\cos 10^\circ+i\sin 10^\circ)$$

We have $z^6=2^6(\cos 60^\circ+i\sin 60^\circ)=32+32i\sqrt{3}$.

Example 2. Raise to the 20th power the number

$$z=\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

The modulus of the number z (Sec. 105) is 1, the argument is -60°. Hence, the modulus of z^{20} is 1 and the argument is $-1200^\circ=-3\cdot 360^\circ-120^\circ$. We thus have

$$z^{20}=\cos(-120^\circ)+i\sin(-120^\circ)=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

Example 3. Find the expression of the cosine and sine of the angle 3φ in terms of the cosine and sine of the angle φ.

Solution. $\cos 3\varphi+i\sin 3\varphi=(\cos\varphi+i\sin\varphi)^3=\cos^3\varphi + 3i\cos^2\varphi\sin\varphi+3i^2\cos\varphi\sin^2\varphi+i^3\sin^3\varphi=\cos^3\varphi -3\cos\varphi\sin^2\varphi+i(3\cos^2\varphi\sin\varphi-\sin^3\varphi)$

Equating abscissas and ordinates (Sec. 99), we find

$$\cos 3\varphi=\cos^3\varphi-3\sin^2\varphi\cos\varphi$$

and

$$\sin 3\varphi=3\cos^2\varphi\sin\varphi-\sin^3\varphi$$

Example 4. In the same way we find

$$\cos 4\varphi=\cos^4\varphi-6\cos^2\varphi\sin^2\varphi+\sin^4\varphi,$$

$$\sin 4\varphi=4\cos^3\varphi\sin\varphi-4\cos\varphi\sin^3\varphi$$

and also the general formulas for $\sin n\varphi$, $\cos n\varphi$ (see Sec. 198).

111. Extracting the Root of a Complex Number

Extracting the root of a number (see Sec. 23, Item 6) is the inverse of raising a number to a power. Therefore (see preceding section), the modulus of a root (of integral index) of a complex number is obtained by extracting that root of the modulus of the radicand, the argument is obtained by dividing the argument by the index of the root:

$$\sqrt[n]{r(\cos\varphi+i\sin\varphi)}=\sqrt[n]{r}\left(\cos\frac{\varphi}{n}+i\sin\frac{\varphi}{n}\right) \qquad \text{(B)}$$

Here the symbol $\sqrt[n]{r}$ denotes a positive number (the principal root of the modulus).

The nth root of any complex number has n distinct values. They all have the same moduli $\sqrt[n]{r}$; the arguments however are obtained from the argument of one of them by successively adding the angle $\frac{1}{n}\cdot 360°$.

Indeed, let φ_0 be the argument of the radicand. Then $\varphi_0+360°$, $\varphi_0+2\cdot 360°$, etc. are also its arguments. Formula (B) shows that for the argument of the root we can take not only $\frac{\varphi_0}{n}$ but also $\frac{\varphi_0}{n}+\frac{1}{n}\cdot 360°$, $\frac{\varphi_0}{n}+\frac{2}{n}\cdot 360°$, and so on. The corresponding values of the root are not all distinct: the argument $\frac{\varphi_0}{n}+\frac{n}{n}\cdot 360°$, i.e., $\frac{\varphi_0}{n}+360°$ yields the same complex number as the argument $\frac{\varphi_0}{n}$; the argument $\frac{\varphi_0}{n}+\frac{n+1}{n}\times 360°=\frac{\varphi_0}{n}+\frac{1}{n}360°+360°$ yields the same complex number as the argument $\frac{\varphi_0}{n}+\frac{1}{n}\cdot 360°$, etc. There will be exactly n distinct values of the root. See examples.

Example 1. Extract the square root of $-9i$. The modulus of this number is 9. Hence the modulus of the root is $\sqrt{9}=3$. The argument of the radicand may be taken equal to $-90°$, $-90°+360°$, $-90°+2\cdot 360°$, etc.

In the first case we obtain

$$(-9i)^{\frac{1}{2}}=\sqrt{9}\,[\cos(-45°)+i\sin(-45°)]=\frac{3}{\sqrt{2}}-\frac{3}{\sqrt{2}}i \qquad (1)$$

In the second case

$$(-9i)^{\frac{1}{2}}=\sqrt{9}(\cos 135^\circ+i\sin 135^\circ)=-\frac{3}{\sqrt{2}}+\frac{3}{\sqrt{2}}i \quad (2)$$

In the third case

$$(-9i)^{\frac{1}{2}}=\sqrt{9}(\cos 315^\circ+i\sin 315^\circ)=\frac{3}{\sqrt{2}}-\frac{3}{\sqrt{2}}i$$

which is the same as in the first. Taking $\varphi=-90^\circ+3\cdot 360^\circ$, $\varphi=-90^\circ+4\cdot 360^\circ$ or $\varphi=-90^\circ-360^\circ$, $-90^\circ-2\cdot 360^\circ$, etc., we alternately get the values (1) and (2).

Example 2. Take the square root of 16. The argument of this number is $360^\circ k$ (k an integer). The argument of the root is $360^\circ k:2=180^\circ k$. If k is zero or an even number, the argument of the root is equal to zero or is a multiple of 360°. Then $16^{\frac{1}{2}}=4(\cos 0^\circ+i\sin 0^\circ)=4$. But if k is an odd number, then the argument will be 180° or will differ from 180° by a multiple of 360°. Then $16^{\frac{1}{2}}=4(\cos 180^\circ+i\sin 180^\circ)=-4$.

Example 3. Extract the cube root of 1. The modulus of the root is $\sqrt[3]{1}=1$. The argument of the radicand is $360^\circ k$ (where k is any integer). The argument of the root is $120^\circ k$. Putting $k=0, 1, 2$, we find three values of the argument of the root: 0°, 120°, 240°. The corresponding values of the root are *:

$$z_1=\cos 0^\circ+i\sin 0^\circ=1,$$
$$z_2=\cos 120^\circ+i\sin 120^\circ=-\frac{1}{2}+\frac{\sqrt{3}}{2}i,$$
$$z_3=\cos 240^\circ+i\sin 240^\circ=-\frac{1}{2}-\frac{\sqrt{3}}{2}i.$$

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* It is useful to check these results. Multiplying the number $z_2=\frac{1}{2}+\frac{\sqrt{3}}{2}i$ by itself by the rule of Sec. 102, we find $z_2^2=-\frac{1}{2}-\frac{\sqrt{3}}{2}i=z_3$. Multiplying once again, we get $z_2^3=z_3z_2=1$. Verification of the root $z_3=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ is the same. Namely,

$$z_3^2=-\frac{1}{2}+\frac{\sqrt{3}}{2}i=z_2,\quad z_3^3=z_2z_3=1$$

In Fig. 17 these values are represented by the points A_1, A_2, A_3. The triangle $A_1A_2A_3$ is an equilateral triangle inscribed in a circle of radius 1.

Example 4. Take the sixth root of -1. The argument of the radicand -1 is $180^\circ + 360^\circ k$. The argument of the

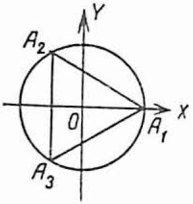

Fig. 17.

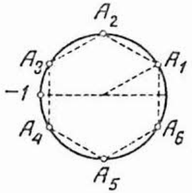

Fig. 18.

root is equal to $30^\circ + 60^\circ k$. We have the following six values of the root:

$$z_1 = \cos 30^\circ + i \sin 30^\circ = \frac{\sqrt{3}}{2} + \frac{1}{2} i,$$
$$z_2 = \cos 90^\circ + i \sin 90^\circ = i,$$
$$z_3 = \cos 150^\circ + i \sin 150^\circ = -\frac{\sqrt{3}}{2} + \frac{1}{2} i,$$
$$z_4 = \cos 210^\circ + i \sin 210^\circ = -\frac{\sqrt{3}}{2} - \frac{1}{2} i,$$
$$z_5 = \cos 270^\circ + i \sin 270^\circ = -i,$$
$$z_6 = \cos 330^\circ + i \sin 330^\circ = \frac{\sqrt{3}}{2} - \frac{1}{2} i$$

The points A_1, A_2, A_3, A_4, A_5, A_6 which represent these values (Fig. 18) are the vertices of a regular hexagon.

From formula (B) it follows that the n roots of some complex number and the n roots of the conjugate of that number are pairwise conjugate.

Example 5. The fourth roots of the number $16(\cos 120^\circ + i \sin 120^\circ) = -8 + 8\sqrt{3}\, i$ are:

$$z_1 = 2(\cos 30^\circ + i \sin 30^\circ) = \sqrt{3} + i,$$
$$z_2 = 2(\cos 120^\circ + i \sin 120^\circ) = -1 + \sqrt{3} i,$$
$$z_3 = 2(\cos 210^\circ + i \sin 210^\circ) = -\sqrt{3} - i,$$
$$z_4 = 2(\cos 300^\circ + i \sin 300^\circ) = 1 - \sqrt{3} i$$

and the fourth roots of the number $16(\cos 120° - i\sin 120°) = -8-8\sqrt{3}\,i$ are

$$\begin{aligned}
\overline{z}_1 &= 2(\cos 30° - i\sin 30°) = \sqrt{3}-i,\\
\overline{z}_2 &= 2(\cos 120° - i\sin 120°) = -1-\sqrt{3}i,\\
\overline{z}_3 &= 2(\cos 210° - i\sin 210°) = -\sqrt{3}+i,\\
\overline{z}_4 &= 2(\cos 300° - i\sin 300°) = 1+\sqrt{3}i
\end{aligned}$$

The numbers z_1 and $\overline{z}_1$, z_2 and $\overline{z}_2$, etc. are conjugate in pairs.

112. Raising a Complex Number to an Arbitrary Real Power

Raising a real number to a fractional power is defined in Sec. 125. However, only real values of the power are considered there. Here, we need a more general definition.

Let it be given by the following formula:

$$[r(\cos\varphi + i\sin\varphi)]^p = r^p(\cos p\varphi + i\sin p\varphi) \qquad \text{(C)}$$

Here, p is any real number and r^p denotes a positive number representing the pth power of the modulus r.

Formula (C) coincides with formula (A) of Sec. 110 when p is integral and with (B) of Sec. 111 when p is the fraction $\frac{1}{n}$. If p is the fraction $\frac{m}{n}$, then by virtue of (C), (A) and (B)

$$[r(\cos\varphi + i\sin\varphi)]^{\frac{m}{n}} = \sqrt[n]{[r(\cos\varphi + i\sin\varphi)]^m} \qquad \text{(D)}$$

which is in agreement with the ordinary definition of a fractional power.

The fractional power of any complex (hence, also real) number has n distinct values (n is the denominator of the fraction). Formula (C) extends also to any irrational exponent p, in which case the pth power of any number has an infinite number of values.

Example 1. Raise the number -16 to the power $\frac{3}{4}$.

We have

$$p = \frac{3}{4}, \quad r = 16, \quad \varphi = 180° + 360°\,k$$

The modulus of the power $(-16)^{\frac{3}{4}}$ is, by (C), $16^{\frac{3}{4}}=8$. The argument of the power is equal to

$$\frac{3}{4}(180^\circ+360^\circ k)=135^\circ+270^\circ k$$

Assuming $k=0, 1, 2, 3$ (the other integral values of k will not yield fresh results), we have the following four values of the power:

$$\begin{aligned}
z_1&=8(\cos 135^\circ+i\sin 135^\circ)=-4\sqrt{2}+4\sqrt{2i},\\
z_2&=8[\cos(135^\circ+270^\circ)+i\sin(135^\circ+270^\circ)]\\
&=8(\cos 45^\circ+i\sin 45^\circ)=4\sqrt{2}+4\sqrt{2i},\\
z_3&=8[\cos(135^\circ+2\cdot 270^\circ)+i\sin(135^\circ+2\cdot 270^\circ)]\\
&=8[\cos(-45^\circ)+i\sin(-45^\circ)]=4\sqrt{2}-4\sqrt{2i},\\
z_4&=8[\cos(135^\circ+3\cdot 270^\circ)+i\sin(135^\circ+3\cdot 270^\circ)]\\
&=8[\cos(-135^\circ)+i\sin(-135^\circ)]=-4\sqrt{2}-4\sqrt{2i}
\end{aligned}$$

These values are represented by the points B_1, B_2, B_3, B_4 (Fig. 19).

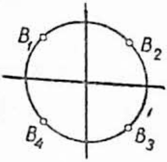

Fig. 19.

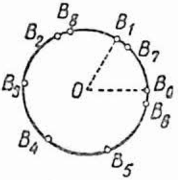

Fig. 20.

Example 2. Raise the number 1 to the power $\frac{1}{2\pi}$. Here, $p=\frac{1}{2\pi}$, $r=1$, $\varphi=360^\circ k$. By (C) we have

$$1^{\frac{1}{2\pi}}=\cos\frac{360^\circ}{2\pi}k+i\sin\frac{360^\circ}{2\pi}k$$

Figure 20 shows the points B_0, B_1, B_2, B_3, . . . depicting the values of the power which result when $k=0, 1, 2, 3$. All lie on a circle of radius 1. No pairs of points coincide. Indeed, each of the angles B_0OB_1, B_1OB_2, etc. is equal to a radian, i. e., each of the arcs B_0B_1, B_1B_2, etc. is one radius in length. If some point B_1 coincided with B_0, this would mean that the circle traversed s times (s a whole number) contained l radii. But then a single circuit of the circle would be of length exactly equal to $\frac{l}{s}$ radii, yet the circumference of a circle is not commensurable with its radius. Hence, no pair of points B_0, B_1, ... can be coincident. The more points we take, the more densely they cover the circle. An infinity of points B accumulate about any point of the circle circumference. And yet on the circumference there are everywhere sites not occupied by any points B. Such, for instance, is the point which is diametrically opposite B_0, or any vertex of any regular polygon in which B_0 is one of the vertices.

Note. It is also possible to define the power of a complex number for a complex exponent. It too has an infinity of values, but the corresponding points do not, in the general case, accumulate. They are spread out.

113. Some Facts about Higher Degree Equations

For general-form equations of third and fourth degree (see Sec. 66), we have formulas which express the roots of the equation in terms of the literal values of the coefficients. These formulas involve radicals of index 2 and 3. They are complicated and too unwieldy for practical use. No such formulas exist for higher-degree equations. It has been proved that it is impossible to express the roots of a general equation of degree higher than fourth in terms of literal coefficients by means of a finite number of additions, subtractions, multiplications, divisions, involutions and evolutions. This is possible only for certain particular types of literal equations of higher degree.

Nevertheless, the roots of any algebraic equation with numerical coefficients can be found in approximate fashion to any desired degree of accuracy.

Prior to the introduction of complex numbers, even a quadratic equation did not always have a solution (see Sec. 92). With the advent of complex numbers, every algebraic equation has at least one root (the coefficients of the algebraic equation may be quite arbitrary, even complex).

An equation of the nth degree cannot have more than n distinct roots, though it may have a smaller number. For instance, the quintic (fifth-degree) equation $(x-3)(x-2)$

$\times(x-1)^3=0$, which in expanded form looks like this: $x^5-8x^4+24x^3-34x^2+23x-6=0$, has the roots $x_1=3$, $x_2=2$, $x_3=1$, and no other roots. Still, it is considered that this equation has five roots: $x_1=3$, $x_2=2$, $x_3=1$, $x_4=1$, $x_5=1$. The root 1 is counted three times because the left member of the equation involves the factor $x-1$ to the third power.

Counting in this manner, we find that every nth degree equation

$$a_0x^n+a_1x^{n-1}+\ldots+a_n=0 \qquad (a_0\neq 0) \tag{1}$$

has exactly n roots. The reason is this. Equation (1) may be represented (uniquely) as

$$a_0(x-x_1)(x-x_2)\ \ldots\ (x-x_n)=0 \tag{2}$$

The numbers $x_1, x_2, \ldots, x_n$ are the roots of (1). There may be, among them, several with the same value (in the last example, we had $x_3=x_4=x_5=1$). This value is counted as a root as many times as it is repeated. If counted in this fashion, the total number of roots is always equal to n.

If the coefficients of an algebraic equation are real and one of the roots is a complex number $a+bi$, then the conjugate complex number $a-bi$ is also a root. For instance, the complex number $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$ is a root of the equation $x^4+1=0$ (Sec. 111); the conjugate complex number $\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$ is also a root of this equation. Thus, an equation with real coefficients always has an even number of complex roots.

Every odd-degree equation with real coefficients has at least one real root (there is always an even number of complex roots and the total number of roots, by hypothesis, is odd).

The sum of the roots of equation (1) is $-\frac{a_1}{a_0}$, while the product of the roots is equal to $(-1)^n\frac{a_n}{a_0}$. These properties were pointed out by the French mathematician Viète in 1591. Viète did not recognize negative numbers (cf. Sec. 67) and so he considered the case when all roots are positive.

Example. The equation $x^5-8x^4+24x^3-34x^2+23x-6=0$ ($n=5$, $a_0=1$, $a_1=-8$, $a_n=-6$) has the following roots (see above): 3, 2, 1, 1, 1. Their sum is 8 $\left(\text{i. e.} -\frac{-8}{1}\right)$ and their product 6 $\left[\text{i. e. } (-1)^5\cdot\frac{-6}{1}\right]$.

These properties (and other similar properties) are derived from a comparison of equations (1) and (2) (all terms in them must be the same, in particular the second and the last).

114. Fundamentals of Inequalities

Two expressions, numerical or literal, connected by the sign $>$ (greater than) or the sign $<$ (less than) form an *inequality* (numerical or literal).

Every true numerical inequality and every literal inequality that is valid for all numerical real values of the letters involved is called an *absolute* or *unconditional inequality*.

Example 1. The numerical inequality $2\cdot 3-5<8-5$ (it is true!) is an unconditional inequality.

Example 2. The literal inequality $a^2>-2$ is unconditional since for any numerical (real) value of a the quantity a^2 is positive or equal to zero and, hence, is always greater than -2.

Two expressions can also be connected by the signs $\leqslant$ (less than or equal to) and $\geqslant$ (greater than or equal to). Thus, the notation $2a\geqslant 3b$ means that the quantity $2a$ is either greater than $3b$ or equal to $3b$. These expressions are also called inequalities.

The literal quantities involved in an inequality can be classified as known and unknown. It is common practice to stipulate which letters are to be taken for the unknown quantities and the known quantities. Ordinarily, the last letters of the alphabet (x, y, z, u, v etc.) are used for unknowns.

To solve an inequality means to indicate the limits within which the real values of the unknown quantities must lie in order for the inequality to be true.

If several inequalities are given, then to solve the system of inequalities means to indicate the limits within which the values of the unknowns must lie so that the given inequalities are true.

Example 3. Solve the inequality $x^2<4$. This inequality is true if $|x|<2$, that is, if x lies between -2 and $+2$. The solution is of the form $-2<x<2$.

Example 4. Solve the inequality $2x>8$. The solution looks like this: $x>4$. Here x is bounded on one side only.

Example 5. The inequality $(x-2)(x-3)>0$ is true if $x>3$ [then both factors $(x-2)$, $(x-3)$ are positive] and also if $x<2$ (then both factors are negative) and is not true when x lies between 2 and 3 (and also when $x=2$ and $x=3$). Therefore, the solution is given by two inequalities:

$$x>3,\ x<2$$

Example 6. The inequality $x^2<-2$ has no solution (cf. Example 2).

115. Basic Properties of Inequalities

1. If $a>b$, then $b<a$; conversely, if $a<b$, then $b>a$.

Example 1. If $5x-1>2x+1$, then $2x+1<5x-1$.

2. If $a>b$ and $b>c$, then $a>c$. In the same way, if $a<b$ and $b<c$, then $a<c$.

Example 2. From the inequalities $x>2y$, $2y>10$ it follows that $x>10$.

3. If $a>b$, then $a+c>b+c$ (and $a-c>b-c$), and if $a<b$, then $a+c<b+c$ (and $a-c<b-c$), which means that *we can add to (or subtract from) both sides of an inequality the same quantity* without changing the sense of the inequality.

Example 3. Given the inequality $x+8>3$. Subtracting 8 from both sides we get $x>-5$.

Example 4. Given the inequality $x-6<-2$. Adding 6 to both sides, we have $x<4$.

4. If $a>b$ and $c>d$, then $a+c>b+d$; in the same way, if $a<b$ and $c<d$, then $a+c<b+d$; that is, *two inequalities having the same sense* (the expression "inequalities of the same sense" means that both inequalities involve the sign $>$ or both have the sign $<$) *may be combined term by term*. This holds true for any number of inequalities, say, if $a_1>b_1$, $a_2>b_2$, $a_3>b_3$, then $a_1+a_2+a_3>b_1+b_2+b_3$.

Example 5. The inequalities $-8>-10$ and $5>2$ are true. Combining them termwise, we find the true inequality $-3>-8$.

Example 6. Given the system of inequalities $\frac{1}{2}x+\frac{1}{2}y<18$, $\frac{1}{2}x-\frac{1}{2}y<4$. Adding them term by term, we get $x<22$.

Note. Two inequalities of the same sense cannot be subtracted termwise one from the other because the result may be either true or untrue. For example if from the inequality $10>8$

we subtract termwise the inequality $2 > 1$, we have the true inequality $8 > 7$, but if we subtract the inequality $6 > 1$ from that inequality, we get an absurd result. Compare with the following item.

5. If $a > b$ and $c < d$, then $a-c > b-d$; if $a < b$ and $c > d$, then $a-c < b-d$; that is, from one inequality it is possible to subtract termwise another inequality of opposite sense* and leave unchanged the sense of the inequality from which the other was subtracted.

Example 7. The inequalities $12 < 20$ and $15 > 7$ are true. Subtracting the first from the second term by term and retaining the sign of the first, we get the true inequality $-3 < 13$. Subtracting the first from the second and leaving the sign of the second, we obtain the true inequality $3 > -13$.

Example 8. Given a system of inequalities: $\frac{1}{2}x + \frac{1}{2}y < 18$, $\frac{1}{2}x - \frac{1}{2}y > 8$. Subtracting the second one from the first, we get $y < 10$.

6. If $a > b$ and m is a positive number, then $ma > mb$ and $\frac{a}{m} > \frac{b}{m}$, or

both sides of an inequality may be divided or multiplied by the same positive number without changing the sense of the inequality.

But if $a > b$ and n is a negative number, then $na < nb$ and $\frac{a}{n} < \frac{b}{n}$, or

both sides of an inequality may be multiplied or divided by the same negative number, but then the sense of the inequality is reversed (it is of course forbidden to multiply and divide both members of an inequality by zero).

Example 9. Dividing both sides of the true inequality $25 > 20$ by 5, we get the true inequality $5 > 4$. But if we divide both sides of $25 > 20$ by -5, we have to reverse the sign $>$ (to $<$) to get the true inequality $-5 < -4$.

Example 10. From the inequality $2x < 12$ it follows that $x < 6$.

Example 11. From the inequality $-\frac{1}{3}x > 4$ it follows that $x < -12$.

Example 12. From the inequality $\frac{x}{k} > \frac{y}{l}$ it follows that $lx > ky$ if the signs of the numbers l and k are the same, and that $lx < ky$ if the signs of the numbers l and k are different.

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* The expression "inequalities of opposite sense" means that one of the inequalities has the sign $>$ and the other, the sign $<$.

116. Some Important Inequalities

1. $|a+b| \leqslant |a|+|b|$. Here, a and b are arbitrary real or complex numbers (but $|a|, |b|$ and $|a+b|$ are always real and positive, see Secs. 69 and 105), that is, *the modulus of a sum does not exceed the sum of the moduli*. Equality occurs only when both numbers a and b have the same argument (Sec. 105), in particular when both numbers are positive or both negative.

Example 1. Let $a=+3$, $b=-5$. Then $a+b=-2$, $|a+b|=2$, $|a|=3$ $|b|=5$. We have $2 < 3+5$.

Example 2. Let $a=4+3i$, $b=6-8i$. Then

$$a+b=10-5i,\ |a+b|=\sqrt{10^2+(-5)^2}=\sqrt{125},$$
$$|a|=\sqrt{4^2+3^2}=5,\ |b|=\sqrt{6^2+(-8)^2}=10,$$
$$|a|+|b|=15$$

We have $\sqrt{125} < 15$.

Note. The inequality $|a+b| \leqslant |a|+|b|$ may be extended to a greater number of terms; thus

$$|a+b+c| \leqslant |a|+|b|+|c|$$

2. $a+\frac{1}{a} \geqslant 2$ (a is a positive number). The equality holds only when $a=1$.

3. $\sqrt{ab} \leqslant \frac{a+b}{2}$ (a and b positive numbers); what this means is that *the geometric mean* (Sec. 59) *of two numbers does not exceed their arithmetic mean*. The equality $\sqrt{ab}=\frac{a+b}{2}$ holds only when $a=b$.

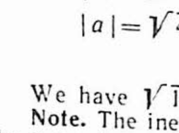

Fig. 21.

Example 3. $a=2$, $b=8$, $\sqrt{ab}=4$, $\frac{a+b}{2}=5$, we have $4 < 5$. This inequality was known 2000 years ago. Its obvious nature is seen geometrically in Fig. 21, where

$$CD=\sqrt{AD\cdot DB} \text{ and } CO=AO=\frac{AD+DB}{2}$$

A generalization of it is the following inequality established by the French mathematician Cauchy in 1821.

4. $\sqrt[n]{a_1a_2\ldots a_n} \leqslant \frac{a_1+a_2+\ldots+a_n}{n}$ (the numbers $a_1, a_2, \ldots, a_n$ are positive). The equality is valid only when all numbers $a_1, a_2, \ldots, a_n$ are equal.

5. $1:\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)\leqslant\sqrt{ab}$ (a and b positive). The sign of equality is valid only when $a=b$.

Example 4. $a=2$, $b=8$, $1:\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{16}{5}$; we have $\frac{16}{5}<4$.

The quantity $1:\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{2ab}{a+b}$ is a mean quantity (Sec. 59) between a and b. It is called the *harmonic mean*.* Thus, the harmonic mean between two quantities does not exceed the arithmetic mean of the quantities. This property can be generalized to any number of quantities; in conjunction with the inequality of Item 4 we have

$$1:\frac{1}{n}\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)\leqslant\sqrt[n]{a_1a_2\ldots a_n}\leqslant\frac{a_1+a_2+\ldots+a_n}{n}$$

6. $\left|\frac{a_1+a_2+\ldots+a_n}{n}\right|\leqslant\sqrt{\frac{a_1^2+a_2^2+\ldots+a_n^2}{n}}$

(the numbers $a_1, a_2, \ldots, a_n$ are arbitrary); thus, the absolute value of the arithmetic mean does not exceed the root-mean-square (Sec. 61). The equals sign holds only when $a_1=a_2=\ldots=a_n$.

Example 5. $a_1=3$, $a_2=4$, $a_3=5$, $a_4=6$.

Here the arithmetic mean is $\frac{a_1+a_2+a_3+a_4}{4}=\frac{9}{2}$ and the root-mean-square is

$$\sqrt{\frac{a_1^2+a_2^2+a_3^2+a_4^2}{4}}=\sqrt{\frac{9+16+25+36}{4}}=\frac{\sqrt{86}}{2}$$

We have $\frac{9}{2}<\frac{\sqrt{86}}{2}$.

$$7.\ a_1b_1+a_2b_2+\ldots+a_nb_n\leqslant\sqrt{a_1^2+a_2^2+\ldots+a_n^2}\times\sqrt{b_1^2+b_2^2+\ldots+b_n^2}$$

the numbers $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$ are arbitrary. The equality is valid only when $a_1:b_1=a_2:b_2=\ldots=a_n:b_n$.

Example 6. Let $a_1=1$, $a_2=2$, $a_3=5$; $b_1=-3$, $b_2=1$, $b_3=2$. We have $a_1b_1+a_2b_2+\ldots+a_nb_n=1\cdot(-3)+2\cdot 1$

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* In the ancient Greek theory of musical harmony an important role was played by the harmonic mean between the lengths of two strings. Whence the name "harmonic".

$+5\cdot 2=9;$

$$\sqrt{a_1^2+a_2^2+\ldots+a_n^2}=\sqrt{1^2+2^2+5^2}=\sqrt{30},$$
$$\sqrt{b_1^2+b_2^2+\ldots+b_n^2}=\sqrt{(-3)^2+1^2+2^2}=\sqrt{14}$$

We have $9<\sqrt{30}\cdot\sqrt{14}$.

8. Chebyshev inequalities. Let the numbers $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n$ be positive.

If $a_1\leqslant a_2\leqslant\ldots\leqslant a_n$ and $b_1\leqslant b_2\leqslant\ldots\leqslant b_n$, then

$$\frac{a_1+a_2+\ldots+a_n}{n}\cdot\frac{b_1+b_2+\ldots+b_n}{n}\leqslant\frac{a_1b_1+a_2b_2+\ldots+a_nb_n}{n} \quad (1)$$

But if $a_1\leqslant a_2\leqslant\ldots\leqslant a_n$ yet $b_1\geqslant b_2\geqslant\ldots\geqslant b_n$, then

$$\frac{a_1+a_2+\ldots+a_n}{n}\cdot\frac{b_1+b_2+\ldots+b_n}{n}\geqslant\frac{a_1b_1+a_2b_2+\ldots+a_nb_n}{n} \quad (2)$$

In both cases, the equality is valid only when all the numbers $a_1, a_2, \ldots, a_n$ are equal and also the numbers $b_1, b_2, \ldots, b_n$ are equal.

Example 7. Let $a_1=1$, $a_2=2$, $a_3=7$ and $b_1=2$, $b_2=3$, $b_3=4$. Then

$$\frac{a_1+a_2+\ldots+a_n}{n}=\frac{1+2+7}{3}=\frac{10}{3},$$
$$\frac{b_1+b_2+\ldots+b_n}{n}=\frac{2+3+4}{3}=3,$$
$$\frac{a_1b_1+a_2b_2+\ldots+a_nb_n}{n}=\frac{1\cdot 2+2\cdot 3+7\cdot 4}{3}=12$$

We have

$$\frac{10}{3}\cdot 3<12$$

Example 8. Let $a_1=1$, $a_2=2$, $a_3=7$ and $b_1=4$, $b_2=3$, $b_3=2$. Then

$$\frac{a_1+a_2+a_3}{3}=\frac{10}{3}, \quad \frac{b_1+b_2+b_3}{3}=3,$$
$$\frac{a_1b_1+a_2b_2+a_3b_3}{3}=8$$

We have

$$\frac{10}{3}\cdot 3>8$$

The inequalities (1) and (2) may be stated thus:

If two sequences of positive quantities have the same number of terms and if in both sequences the terms do not diminish (or do not increase), then the product of the arithmetic means does not exceed the arithmetic mean of the products. But if in one sequence the terms do not decrease and in the other they do not increase, then the opposite inequality is valid.

These inequalities were discovered in 1886 by the celebrated Russian mathematician P. L. Chebyshev (1821—1894). He also generalized them and proved the following inequalities:

If $0 < a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$ and $0 < b_1 \leqslant b_2 \leqslant \ldots \leqslant b_n$, then

$$\sqrt{\frac{a_1^2+a_2^2+\ldots+a_n^2}{n}}\sqrt{\frac{b_1^2+b_2^2+\ldots+b_n^2}{n}} \leqslant \sqrt{\frac{(a_1b_1)^2+(a_2b_2)^2+\ldots+(a_nb_n)^2}{n}}, \tag{3}$$

$$\sqrt[3]{\frac{a_1^3+a_2^3+\ldots+a_n^3}{n}}\sqrt[3]{\frac{b_1^3+b_2^3+\ldots+b_n^3}{n}} \leqslant \sqrt[3]{\frac{(a_1b_1)^3+(a_2b_2)^3+\ldots+(a_nb_n)^3}{n}} \tag{4}$$

and so on.

However if $0 < a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$ but $b_1 \geqslant b_2 \ldots \geqslant b_n > 0$, then the opposite inequalities hold true.

117. Equivalent Inequalities. Basic Techniques for Solving Inequalities

Two inequalities involving the same unknowns are called *equivalent* if they are true for the same values of the unknowns.

The equivalence of two systems of inequalities is defined in the same manner.

Example 1. The inequalities $3x+1>2x+4$ and $3x>2x+3$ are equivalent since both are true for $x>3$ and both are untrue when $x \leqslant 3$.

Example 2. The inequalities $2x \leqslant 6$ and $x^2 \leqslant 9$ are not equivalent since the solution of the former is $x \leqslant 3$, while the solution of the latter is $-3 \leqslant x \leqslant 3$, so that, say for $x=-4$, the former is true and the latter untrue.

The process of solving an inequality consists, in the main, in replacing the given inequality (or the given system of inequalities) by other, equivalent, inequalities (see Sec. 215 for a graphical solution of inequalities). When solving inequalities use the following basic techniques (cf. Sec. 82).

1. Replacing one expression by an identical expression.

2. Transposing a term from one side of the inequality to the other with a reversal of sign (by virtue of Sec. 115, Item 3).

3. Multiplying or dividing both sides of an inequality by the same numerical quantity (not equal to zero). If the multiplier is positive, then the sense of the inequality remains unchanged, if it is negative, the sense of the inequality is reversed (Sec. 115, Item 6).

Each one of these transformations yields an inequality that is equivalent to the original one.

Example 3. Given the inequality $(2x-3)^2 < 4x^2+2$. Replace the left member by the identical expression $4x^2-12x+9$. We get $4x^2-12x+9<4x^2+2$. Transpose $4x^2$ to the left member and 9 to the right member. Collecting like terms, we get $-12x<-7$. Divide both sides of the inequality by -12; this necessitates reversing the sense of the inequality. The solution of the given inequality is $x > \frac{7}{12}$.

To multiply (and of course divide) an inequality by zero is not permissible. When multiplying or dividing both members of an inequality by literal expressions, we get an inequality which, as a rule, is not equivalent to the original one.

Example 4. Given the inequality $(x-2)x < x-2$.

Divide both sides by $x-2$ to get $x < 1$. But this inequality is not equivalent to the original one because, for instance, the value $x=0$ does not satisfy the inequality $(x-2)x < x-2$. Again, the inequality $x > 1$ is not equivalent to the original one because, for example, the value $x=3$ does not satisfy the inequality $(x-2)x < x-2$.

118. Inequalities Classified

Inequalities involving unknown quantities may be divided into *algebraic* and *transcendental inequalities*. Algebraic inequalities are further subdivided into *inequalities of first, second*, etc. *degree*. This classification is the same as that for equations (Sec. 83).

Example 1. The inequality $3x^2-2x+5>0$ is an algebraic inequality of the second degree.

Example 2. The inequality $2^x > x+4$ is a transcendental inequality.

Example 3. The inequality $3x^2-2x+5>3x(x-2)$ is an algebraic inequality of the first degree because it can be reduced to the inequality $4x+5>0$.

119. Inequalities of the First Degree in One Unknown

A first-degree (linear) inequality in one unknown can be reduced to the form

$$ax > b$$

It has the solution

$$x > \frac{b}{a} \quad \text{if} \quad a > 0$$

and

$$x < \frac{b}{a} \quad \text{if} \quad a < 0$$

Example 1. Solve the inequality $5x-3>8x+1$.

Solution. $5x-8x>3+1$, $-3x>4$, $x<-\frac{4}{3}$.

Example 2. Solve the inequality $5x+2<7x+6$.

Solution. $5x-7x<6-2$, $-2x<4$, $x>-2$.

Example 3. Solve the inequality $(x-1)^2<x^2+8$.

Solution. $x^2-2x+1<x^2+8$, $-2x<7$, $x>-\frac{7}{2}$.

Note. An inequality of the form $ax+b>a_1x+b_1$ is an inequality of the first degree if a and a_1 are distinct. Otherwise the inequality is reduced to a numerical (true or untrue) inequality.

Example 4. Given the inequality $2(3x-5)<3(2x-1)+5$. It is equivalent to the inequality $6x-10<6x+2$ and the latter can be reduced to the numerical (unconditional) inequality $-10<2$. Hence, the original inequality is unconditional.

Example 5. The inequality $2(3x-5)>3(2x-1)+5$ reduces to the meaningless numerical inequality $-10>2$. Hence, the original inequality has no solutions.

120. A System of Inequalities of the First Degree

To solve a system of first-degree inequalities, find the solution of each inequality separately and compare the solutions. This comparison either yields the solution of the system or reveals that the system does not have any solutions.

Example 1. Solve the system of inequalities

$$4x-3>5x- \;, \quad 2x+4<8x$$

The solution of the first inequality is $x < 2$; of the second, $x > \frac{2}{3}$. The solution of the system is $\frac{2}{3} < x < 2$.

Example 2. Solve the system of inequalities

$$2x-3 > 3x-5, \quad 2x+4 > 8x$$

For the first we have the solution $x < 2$; for the second, $x < \frac{2}{3}$. The solution of the system is $x < \frac{2}{3}$ (under this condition the inequality $x < 2$ is certainly true).

Example 3. Solve the system of inequalities

$$2x-3 < 3x-5, \quad 2x+4 > 8x$$

The solution of the first inequality is $x > 2$, the solution of the second, $x < \frac{2}{3}$. These conditions are contradictory. The system has no solutions.

Example 4. Solve the system of inequalities

$$2x < 16, \quad 3x+1 > 4x-4, \quad 3x+6 > 2x+7, \quad x+5 < 2x+6$$

The solutions are, respectively, $x < 8$, $x < 5$, $x > 1$, $x > -1$. Comparing these conditions we find that the first two may be replaced by the second alone, and the third and fourth, by the third alone. The solution of the system is $1 < x < 5$.

121. Elementary Inequalities of the Second Degree in One Unknown

1. The inequality $x^2 < m$. (1)

(a) If $m > 0$, the solution is

$$-\sqrt{m} < x < \sqrt{m} \tag{1a}$$

(b) If $m \leqslant 0$, then there is no solution (the square of a real number cannot be negative)

2. The inequality $x^2 > m$. (2)

a) If $m > 0$, then (2) is valid, firstly, for all values of x greater than $\sqrt{m}$, and, secondly, for all values of x less than $-\sqrt{m}$:

$$x > \sqrt{m} \quad \text{or} \quad x < -\sqrt{m} \tag{2a}$$

(b) If $m=0$, then (2) holds true for all x except $x=0$:

$$x > 0 \quad \text{or} \quad x < 0 \tag{2b}$$

(c) If $m < 0$, then (2) is an unconditional inequality.

Example 1. The inequality $x^2 < 9$ has the solution $-3 < x < 3$.

Example 2. The inequality $x^2 < -9$ has no solutions.

Example 3. The inequality $x^2 > 9$ has for a solution the set of all numbers greater than 3 and the set of all numbers less than -3.

Example 4. The inequality $x^2 > -9$ is unconditional.

122. Inequalities of the Second Degree in One Unknown (General Case)

Dividing a second-degree inequality by the coefficient of x^2, we reduce it to one of the following types:

$$x^2 + px + q < 0, \tag{1}$$

$$x^2 + px + q > 0 \tag{2}$$

Transpose the constant term to the right side and add to both sides $\left(\frac{p}{2}\right)^2$. This yields, respectively,

$$\left(x + \frac{p}{2}\right)^2 < \left(\frac{p}{2}\right)^2 - q, \tag{1'}$$

$$\left(x + \frac{p}{2}\right)^2 > \left(\frac{p}{2}\right)^2 - q \tag{2'}$$

If we denote $x + \frac{p}{2}$ by z and $\left(\frac{p}{2}\right)^2 - q$ by m, we get the elementary inequalities

$$z^2 < m, \tag{1''}$$

$$z^2 > m \tag{2''}$$

The solution of these inequalities was given in the preceding section. Knowing it, we find the solution of (1) or (2).

Example 1. Solve the inequality $-2x^2 + 14x - 20 > 0$. Divide both members by -2 (Sec. 117, Item 3) to get $x^2 - 7x + 10 < 0$. Transposing the constant term 10 to the right and adding $\left(\frac{7}{2}\right)^2$ to both members, we get $\left(x - \frac{7}{2}\right)^2 < \frac{9}{4}$, whence (Sec. 121, Case 1a)

$$-\frac{3}{2} < x - \frac{7}{2} < \frac{3}{2}$$

Adding $\frac{7}{2}$, we find $-\frac{3}{2} + \frac{7}{2} < x < \frac{3}{2} + \frac{7}{2}$, that is, $2 < x < 5$.

Example 2. Solve the inequality $-2x^2+14x-20<0$. Performing the same transformations, we get the inequality $\left(x-\frac{7}{2}\right)^2>\frac{9}{4}$, whence (Sec. 121, Case 2a) we find that our inequality is valid, firstly, for $x-\frac{7}{2}>\frac{3}{2}$; that is, for $x>5$, and, secondly, for $x-\frac{7}{2}<-\frac{3}{2}$, or for $x<2$.

Example 3. Solve the inequality $x^2+6x+15<0$. Transposing the constant term to the right and adding to both members $\left(\frac{6}{2}\right)^2$, i.e., 9, we find $(x+3)^2<-6$. This inequality (Sec. 121, Case 1b) has no solutions and so the given inequality has no solutions.

Example 4. Solve the inequality $x^2+6x+15>0$. As in Example 3, we find $(x+3)^2>-6$. This inequality (Sec. 121, Case 2c) is unconditional.

123. Arithmetic Progressions

At one time the term progression was used in mathematics to denote any sequence of numbers generated by some law that permitted extending the sequence indefinitely in one direction. For example, by squaring the sequence of whole numbers, we get the sequence 1, 4, 9, 16, 25, etc. This can be continued without end simply by applying the law of formation of the sequence. The numbers thus generated are called the *terms* of the sequence. At the present time the term "progression" is largely confined to the two most important kinds of number sequences: arithmetic and geometric, all other progressions of numbers being termed *sequences.*

An arithmetic progression is a sequence of numbers such that the difference between any two successive terms is a constant called the *common difference.*

Example 1. The sequence of natural numbers 1, 2, 3, 4, 5, ... is an arithmetic progression with common difference 1.

Example 2. The sequence of numbers 10, 8, 6, 4, 2, 0, —2, —4 ... is an arithmetic progression with common difference —2.

Any term of an arithmetic progression may be computed by the formula

$$a_n=a_1+d\,(n-1)$$

where a_1 is the first term of the progression, d is the common difference, and a_n is the nth term, n being the number of the term.

The sum of the first n terms of an arithmetic progression is given by the formula

$$s_n = \frac{(a_1 + a_n)\cdot n}{2}$$

Example 3. In the progression 12, 15, 18, 21, 24, ... the tenth term is egual to $a_{10} = 12 + 3\cdot 9 = 39$.

The sum of the first ten terms is

$$s_{10} = \frac{(a_1 + a_{10})\,10}{2} = \frac{(12+39)\,10}{2} = 255$$

Example 4. The sum of all integers from 1 to 100 inclusive is $\frac{(1+100)\,100}{2} = 5050$.

124. Geometric Progressions

A geometric progression is a sequence of numbers such that the ratio of each term to the immediately preceding one is a constant called the *common ratio* (or, simply, *ratio*).

Example 1. The numbers 5, 10, 20, 40, ... form a geometric progression with ratio 2.

Example 2. The numbers 1, 0.1, 0.01, 0.001, etc. constitute a geometric progression with ratio 0.1.

A geometric progression is termed *increasing* when the absolute value of its ratio is greater than unity (as in Example 1) and *decreasing* when it is less than unity (as in Example 2).

Note. The ratio of a progression may be a negative number, but such progressions are of no practical significance.

Any term of a geometric progression may be computed from the formula

$$a_n = a_1 q^{n-1} \tag{1}$$

where a_1 is the first term, q is the ratio, and a_n is the nth term, n being the number of the term.

The sum of the first n terms of a geometric progression (whose ratio is not unity) is given by the formula

$$s_n = \frac{a_n q - a_1}{q - 1} = \frac{a_1 - a_n q}{1 - q} \tag{2}$$

The first expression is most conveniently used for increasing progressions, the second for decreasing progressions.

But if $q=1$, then the progression consists of equal terms and in place of (2) we have $s_n = na_1$.

Example 3. In the geometric progression, 5, 10, 20, 40, ... the tenth term $a_{10} = 5 \cdot 2^9 = 5 \cdot 512 = 2560$. The sum of the first ten terms is

$$s_{10} = \frac{a_{10} \cdot 2 - a_1}{2-1} = 5115$$

The sum of an infinitely decreasing progression is a number approached without bound by the sum of the first n terms of the decreasing progression when the number n increases without bound.

The sum of an infinitely decreasing progression is given by the formula

$$s = \frac{a_1}{1-q}$$

Example 4. The sum of the infinite geometric progression $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \left(a_1 = \frac{1}{2},\ q = \frac{1}{2}\right)$ is $\dfrac{\frac{1}{2}}{1-\frac{1}{2}} = 1$ that is, the sum $\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}$ approaches the number 1 without bound as n increases without bound.

125. Negative, Zero and Fractional Exponents

Raising a quantity to the nth power was originally understood as an n-fold repetition of a certain number as a factor. From this point of view, such expressions as 9^{-2} or $9^{1\frac{1}{2}}$ appear to be meaningless since it is clearly impossible to take 9 as a factor for a total of minus two times or one and one half times. Nevertheless, mathematicians attach a very definite meaning to such expressions; namely, 9^{-2} is considered to be equal to $\frac{1}{9^2} = \frac{1}{81}$; $9^{1\frac{1}{2}}$, to $\sqrt{9^3} = (\sqrt{9})^3 = 27$, etc. Here again we encounter the generalization of the concept of a mathematical operation that is constantly going on in mathematics. The simplest and earliest generalization of this kind was that of the operation of multiplication to the case of a fractional factor (see Sec. 34). It is possible to get along without introducing fractional or negative exponents, but then problem of a single kind would have to be solved by a multitude of diverse rules instead of one. The problems we have in mind are to be found mostly in higher mathematics and therefore many concrete examples are beyond the scope of this book. However, one of these problems is studied in detail in elementary mathematics. It has to do with logarithms (see Sec. 126). It is worth noting that the theory of logarithms, which today is intimately bound up with the generalization of the concept of a power, dispensed with fractional and negative exponents for a whole century after its discovery (at the turn of the 17th century); the same goes for the problems of higher mathematics that we mentioned. It was only at the end of the 17th century that the number and complexity of mathematical problems urgently called for a generalization of the concept of a power. That was the path taken by certain scholars and, most notably, Newton.

Definition of negative power.* By definition, the power of any number with a negative (integral) exponent is unity divided by the power of that number with a positive exponent, the value of which is equal to the absolute value of the negative exponent; or

$$a^{-m}=\frac{1}{a^m}$$

Examples. $2^{-3}=\frac{1}{2^3}=\frac{1}{8}$, $\left(\frac{3}{4}\right)^{-2}=1:\left(\frac{3}{4}\right)^2=\frac{16}{9}$,

$$(-4)^{-3}=1:(-4)^3=-\frac{1}{64}\text{, etc.}$$

The equation $a^{-m}=1:a^m$ holds true both for positive m and for negative m. If, say, $m=-5$, then $-m$ will be equal to $+5$ and our formula will look like this: $a^5=\frac{1}{a^{-5}}$, which is in agreement with the definition given above.

Operations involving negative exponents obey all the rules that hold for positive exponents. What is more, it was only after the introduction of negative exponents that the rules for handling positive exponents acquired full generality.

Thus, the formula $a^m:a^n=a^{m-n}$ (see Sec. 89) can now be applied not only when $m>n$, but also when $m<n$.

Example. $a^5:a^8=a^{5-8}=a^{-3}$. Indeed, according to the definition $a^{-3}=\frac{1}{a^3}$ so that the expression $a^5:a^8=a^{-3}$ means

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* The terms "negative power", "zeroth power", and "fractional power" are taken to mean powers with a negative, zero and fractional exponent, respectively.

$\frac{a^5}{a^8}=\frac{1}{a^3}$. For the formula $a^m:a^n=a^{m-n}$ to have generality, it must hold true when $m=n$ as well. For this purpose we make the following definition.

Definition of a zeroth power. *The zeroth power of any nonzero number is unity* [the expression 0^0, like the expression $\frac{0}{0}$ (see Sec. 37), is indeterminate].

Examples. $3^0=1$, $(-3)^0=1$, $\left(-\frac{2}{3}\right)^0=1$, $a^5:a^5=a^0=1$.

Definition of a fractional power. *To raise a* (real) *number* a *to the power* $\frac{m}{n}$ *means to extract the nth root of the mth power of* a. The fractional powers of complex numbers are discussed in Sec. 112.

$$\text{Examples. } 9^{\frac{3}{2}}=\sqrt{9^3}=27,\ \left(\frac{8}{27}\right)^{1\frac{1}{3}}=\left(\frac{8}{27}\right)^{\frac{4}{3}}=\sqrt[3]{\left(\frac{8}{27}\right)^4}$$
$$=\frac{16}{81},\quad 3^{2\frac{1}{2}}=3^{\frac{5}{2}}=\sqrt{243}\approx 15.58.$$

Note 1. The base a could be taken negative, but then its fractional powers might not result in real numbers. For instance,

$$(-2)^{\frac{3}{4}}=\sqrt[4]{(-2)^3}=\sqrt[4]{-8}$$

The root $\sqrt[4]{-8}$ cannot be a real number.

Ordinarily, elementary mathematics considers only positive bases of fractional powers.

Note 2. The exponents however, may be positive, negative or fractional. Negative exponents are of no less importance than positive exponents. In order to master logarithmic computations, it is necessary to do as many exercises as possible in order to get a proper understanding of negative and fractional exponents.

$$\text{Examples. } 9^{-\frac{3}{2}}=1:9^{\frac{3}{2}}=\frac{1}{27},\quad \left(\frac{8}{27}\right)^{-1\frac{2}{3}}=1:\left(\frac{8}{27}\right)^{1\frac{2}{3}}$$
$$=\frac{243}{32},\ 3^{-2\frac{1}{2}}=1:3^{2\frac{1}{2}}=\frac{1}{\sqrt{243}}\approx 0.0642.$$

The introduction of fractional exponents does not involve any changes in the laws of exponents. Thus, the formula $a^m \cdot a^n = a^{m+n}$ and others are still valid.

Example. $a^{\frac{5}{7}} \cdot a^{-\frac{3}{7}} = a^{\frac{2}{7}}$. True enough. $a^{\frac{5}{7}} = \sqrt[7]{a^5}$; $a^{-\frac{3}{7}} = 1 : \sqrt[7]{a^3}$, $a^{\frac{2}{7}} = \sqrt[7]{a^2}$ so that our notation signifies $\sqrt[7]{a^5} \cdot \frac{1}{\sqrt[7]{a^3}} = \sqrt[7]{a^2}$ which is true (see Sec. 90, Rule 4).

126. The Method of Logarithms. Construction of Logarithmic Tables

The operations of multiplication, division, involution and evolution are much more labour-consuming than those of addition and subtraction, especially when they involve multidigit numbers. An insistent need for such operations arose in the 16th century in connection with the development of sea navigation which gave rise to improved astronomical observations and calculations. It was astronomical calculations that gave rise, at the turn of the 17th century, to computations by means of logarithms.

Today, such calculations are used whenever one has to do with large numbers. They are already useful when dealing with four-digit numbers and are absolutely necessary when five-place accuracy is required. Greater accuracy is rarely needed in practical situations.

The value of the logarithmic method consists in reducing multiplication and division of numbers to addition and subtraction which are much easier to perform. Likewise, involution and evolution and also a number of other computations (trigonometric, for instance) are greatly simplified.

The idea of the method can be illustrated in a few examples.

Suppose we have to multiply 10,000 by 100,000. The ordinary scheme of multiplying long numbers is not needed here at all: simply count the number of zeros in the multiplicand (4) and the multiplier (5) and add these numbers to get 9, which is the number of zeros in the product: 1,000,000,000 (9 zeros) This computation is legitimate because the factors are (integral) powers of 10: we multiply

10^4 by 10^5 and the exponents are added. In the same way we perform division of powers of ten (division in this case is replaced by subtraction of the exponents).

But only a few numbers can be divided and multiplied in this manner. In the first million (excluding 1) we have only 6 such numbers, 10, 100, 1000, 10,000, 100,000, 1,000,000. Now there would be more numbers involved in multiplication and division if in place of the base 10 we took, say, 2, or some base closer to 1. For 2 as a base, we construct a table of the first 12 powers:

Exponent (or logarithm):	1	2	3	4	5	6	7	8	9	10	11	12
Power (or number):	2	4	8	16	32	64	128	256	512	1024	2048	4096

We will now use the term *logarithm* for the numbers (exponents) in the upper row, and the term *number* for the numbers (powers) in the lower row.

To multiply any two numbers of the lower row, add the two numbers above them in the upper row. To illustrate, we find the product of 32 by 64 by adding the 5 above 32 to the 6 above 64 to get 11. The answer is under 11: 2048. To divide 4096 by 256, take the numbers 12 and 8 above them; subtract 8 from 12 to get 4. The answer is under the number 4: 16. If we continue the table to the left and introduce zero and negative powers of the number 2, it is possible to perform division of small numbers by larger numbers.

Although there are fewer gaps between the powers of 2 than between the powers of 10, this table still involves only a few numbers and so is of little practical value. But if we take for the base a number much closer to 1 than the number 2, this defect will be overcome.

Let us take for the base the number 1.00001. There will be over a million (1,151, 292) successive powers of this number between 1 and 100,000. If we round off the values of these powers and retain only 6 significant digits, then a million rounded results will include all the whole numbers from 1 to 100,000. True, these will only be approximate values of the powers. But since in the multiplication and division of five-digit integers we are interested only in the first five places of the result, such tables will enable us to multiply, divide, etc. five-digit integers and, consequently, decimals with five significant digits.

That is precisely how the first tables of logarithms were constructed. * Their computation required many years of arduous labour. Today, anyone could do the job in about a month using the methods of higher mathematics. Three hundred years ago it required a lifetime. But the result was that many thousands of computors calculated many times faster through the use of these tables which were constructed once and for all time.

At the present time, logarithmic tables use the base 10 because this yields a number of computational advantages since our system of numeration is decimal. To obtain whole numbers it is necessary to use fractional powers of the number 10.

The logarithm of some number to the base 10 is called the *common logarithm*. Compilation of tables of common logarithms does not involve any particular difficulties if a table to the base 1.00001 has already been compiled. Indeed, suppose we want to find the common logarithm of the number 3, that is the exponent of the power to which we have to raise 10 in order to obtain 3. In the table to the base 1.00001 we find

$$10 \approx 1.00001^{230,258}$$

$$3 \approx 1.00001^{109,861}$$

Raising both members of the first equality to the power $\frac{1}{230,258}$, we get $1.00001 \approx 10^{(1:230,258)}$ and so the second equality can be written as $3 \approx 10^{(109,861:230,258)}$, which is to say that the logarithm of the number 3 to the base 10 is 0.47712. In the same way we can find the common logarithms of other numbers. **

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* In about 1590 by Bürgi of Switzerland. Independently of him, and somewhat later the Scotsman Napier constructed a table based on a number very close to unity, but less than unity. Bürgi published his work in the year 1620, whereas Napier's tables appeared earlier, in 1614.

** The idea of constructing of table of logarithms to the base 10 belongs to the Scotsman Napier and the Englishman Briggs. Together they undertook to recalculate the earlier tables of Napier to the new base of 10. After the death of Napier, Briggs continued the work and published it completely in 1624. That is why the base—10 (common) logarithms are also called Briggsian. Fractional powers were not yet used in mathematics, but Napier and Briggs did without them since they defined the term logarithm somewhat differently from the presently accepted definition.

127. Basic Properties of Logarithms

The logarithm of a number N to a base a is the exponent x indicating the power to which a must be raised to obtain N.

Notation: $\log_a N = x$. Symbolically, $\log_a N = x$ is equivalent to $a^x = N$.

Examples. $\log_2 8 = 3$ since $2^3 = 8$; $\log_{1/2} 16 = -4$ since $\left(\frac{1}{2}\right)^{-4} = 16$; $\log_{1/2}\left(\frac{1}{8}\right) = 3$ since $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$.

From the definition of a logarithm follows the identity

$$a^{\log_a N} = N$$

Examples. $2^{\log_2 8} = 8$, i. e., $2^3 = 8$; $5^{\log_5 25} = 25$, $10^{\log_{10} N} = N$.

The numbers a (the *logarithmic base*) and N (the *number*) may be taken integral and fractional (see examples), but they must be positive if we want the logarithms to be real numbers.

The logarithms themselves may be negative. *Negative logarithms are just as important, practically, as positive logarithms.*

If for the base we take a number greater than unity (say, 10), then the larger number has a larger logarithm. The logarithms of numbers greater than unity are positive, those of numbers less than unity are negative. The logarithm of unity to any base is zero. The logarithm of a number equal to the base is always unity (in the common logarithms, $\log 10 = 1$). *

The logarithm of a product is equal to the sum of the logarithms of the factors:

$$\log_a (pq) = \log_a p + \log_a q$$

The logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor:

$$\log_a \frac{p}{q} = \log_a p - \log_a q$$

The logarithm of the power of a number is equal to the exponent times the logarithm of the number:

$$\log_a p^m = m \log_a p$$

---

* The number a must not equal unity, otherwise numbers not equal to unity will not have a logarithm and any number will be the logarithm of unity.

We use the symbol $\log x$ to mean $\log_{10} x$ [translator].

The logarithm of a root of a number is equal to the logarithm of the number divided by the index of the root:

$$\log_a \sqrt[m]{p} = \frac{1}{m} \log_a p$$

$\left(\text{this is a consequence of the preceding property since } \sqrt[m]{p} = p^{\frac{1}{m}}\right)$.

Warning. The logarithm of a sum is not equal to the sum of the logarithms; it is incorrect to write $\log_a p + \log_a q$ in place of $\log_a (p+q)$. This is a common mistake.

To take the logarithms of an expression means to express its logarithm in terms of the logarithms of the quantities that make up the expression.

Examples of taking logarithms:

$$(1)\ \log_a \frac{2p^2q}{\sqrt[3]{m^2}} = \log_a (2p^2qm^{-2/3})$$

$$= \log_a 2 + 2\log_a p + \log_a q - \frac{2}{3}\log_a m$$

$$(2)\ x = \frac{14.352 \cdot \sqrt{0.20600}}{185.06 \cdot 43110^2},$$

$$\log x = \log 14.352 + \frac{1}{2}\log 0.20600 - \log 185.06 - 2\log 43{,}110$$

Using a table of common logarithms, find $\log 14.352$, $\log 0.20600$, etc. and compute the right member of our equation; this is $\log x$. Then, using the table, find the number x from its logarithm. For more details, see Secs. 131-134.

128. Natural Logarithms. The Number e

For practical purposes the most convenient are logarithms to the base 10 (common logarithms). In theoretical investigations however, it is more convenient to use a different base, namely the irrational number $e = 2.71828183$ (to eight decimal places). This amazing, at first glance, fact can only be explained in higher mathematics. Here we will merely show where the number came from. It is closely connected with the mode of computing logarithms that was explained in Sec. 126. When for the base we take a number $1 + \frac{1}{n}$

close to unity, say 1.00001 ($n = 100{,}000$), then for small numbers we get enormous logarithms such as 109,861 for the number 3. To make this logarithm of the same order of magnitude as 3, it has to be reduced by a factor of $n = 100{,}000$. Then it will be 1.09861. The number 3 will have the logarithm 1.09861 if for the base we take

$$\left(1+\frac{1}{n}\right)^n = 1.00001^{100{,}000} \text{ and not } 1+\frac{1}{n} = 1.00001$$

Indeed, we have

$$3 = (1.00001)^{109{,}861} = 1.00001^{100{,}000 \cdot 1.09861}$$
$$= (1.00001^{100{,}000})^{1.09861}$$

Computing the quantity $1.00001^{100{,}000}$ to eight places of decimals, we get

$$\left(1+\frac{1}{n}\right)^n = 2.71826763 \quad (n = 100{,}000)$$

This number is very close to the number e; the first five digits coincide. If we took a number still closer to unity, say 1.000001 ($n = 1{,}000{,}000$), then reasoning as before, we would see that

$$\left(1+\frac{1}{n}\right)^n = 1.000001^{1{,}000{,}000}$$

is a still more convenient base.

To eight places of decimals, this number is 2.71828047. The first six digits are the same as those in the number e and the seventh digit differs only by unity. The greater the number n, the less the number $\left(1+\frac{1}{n}\right)^n$ differs from the number e. In other words, *the number e is the limit to which $\left(1+\frac{1}{n}\right)^n$ tends as n increases without bound*. That is the definition of the number e.

We have seen that the base $1+\frac{1}{n}$ and, hence, $\left(1+\frac{1}{n}\right)^n$ as well, enables us to compute logarithms of all possible numbers the more exactly, the greater the number n. It is natural to expect that for this purpose the most convenient number is the limit to which $\left(1+\frac{1}{n}\right)^n$ tends as n increases without bound, which is the number e. That precisely is the case. Computation of logarithms to the base e can be performed more quickly than to any other base. The methods for such calculation are given in higher mathematics.

The number e itself can be expressed as a decimal to any desired degree of accuracy. Some tables contain approximate values of e that far exceed any practical demands. However, it is impossible to express the number e exactly by any decimal fraction or any rational fraction. What is more, e is not only irrational, it is transcendental (see Sec. 91).

Logarithms taken to the base e are called *natural logarithms*. Sometimes they are called Napierian, but this is wrong historically.*

Notation. Natural logarithms are usually denoted by $\ln x$ instead of $\log_e x$.

Example. $\ln 3 = 1.09861$.

To find the natural logarithm of a number N from its common logarithm, divide the common logarithm of N by the common logarithm of e (which is equal to $0.43429\ldots$):

$$\ln N = \frac{\log N}{\log e} \approx \frac{\log N}{0.43429} \approx 2.30259 \log N$$

The quantity $\log e = 0.43429$ is called the *modulus of common* logarithms with respect to natural logarithms and is denoted by M, so that

$$\ln N = \frac{1}{M} \log N \; {}^{**}$$

---

* The base actually used by Napier was the number $1-0.0000001$. If we wanted to reduce all logarithms of Napier's table by a factor of $10{,}000{,}000 = 10^7$ (see example analyzed above), we would have to take as the base the number $\left(1-\frac{1}{k}\right)^k$, where $k=10^7$, which could then be called the base of Napier's table. But this number is by no means equal to e $\left(\text{it differs very slightly from the number } \frac{1}{e}\right)$.

** The rules given here for converting from natural logarithms to common logarithms and vice versa are particular cases of the general formulas

$$\log_a N = \log_b N \cdot \log_a b, \quad \log_a N = \frac{\log_b N}{\log_b a}$$

which permit converting from the logarithm of a number N to the base b to a logarithm of the same number to the base a. The second formula yields for $N=b$

$$\log_a b = \frac{1}{\log_b a}$$

Example. From the table of common logarithms we have $\log 2 = 0.30103$, whence

$$\ln 2 = \frac{1}{M} \cdot 0.30103 = 0.69315$$

To find the common logarithm of a number N from a given natural logarithm of N, multiply the natural logarithm by the modulus of common logarithms with respect to natural logarithms: $M = \log e$:

$$\log N = \log e \ln N = M \ln N \approx 0.43429 \ln N$$

Example. $\ln 3 = 1.09861$, whence $\log 3 = M \cdot 1.09861 = 0.47712$.

Tables are provided to simplify multiplication by M and $\frac{1}{M}$. They contain products of M and $\frac{1}{M}$ by all one-digit or all two-digit factors. We give below a table for multiplication of M and $\frac{1}{M}$ by single-digit numbers.

	Multiples of M	Multiples of $\frac{1}{M}$
1	0.43429	2.30259
2	0.86859	4.60517
3	1.30288	6.90776
4	1.73718	9.21034
5	2.17147	11.51293
6	2.60577	13.81551
7	3.04006	16.11810
8	3.47436	18.42068
9	3.90865	20.72327

129. Common Logarithms

From now on, we shall simply use the word logarithm in the meaning of common logarithm.

The logarithm of unity is zero.

The logarithms of 10, 100, 1000, etc., are 1, 2, 3, and so on, that is, they have as many positive units as there are zeros following unity.

The logarithms of the numbers 0.1, 0.01, 0.001, etc. are equal to −1, −2, −3, etc.; hence they have as many negative units as there are zeros (including the zero in the units place) preceding unity.

The logarithms of the other numbers have a fractional part which is called the *mantissa*. The integral (whole-number) portion of the logarithm is termed the *characteristic*.

Numbers greater than unity have positive logarithms. Positive numbers less than unity have negative logarithms (negative numbers do not have real logarithms).

For example, * $\log 0.5 = -0.30103$, $\log 0.005 = -2.30103$.

For the sake of convenience in locating the logarithm from a number or the number from the logarithm, negative logarithms are not given in this natural form but in an "artificial" form. In the so-called artificial form, a negative logarithm *has a positive mantissa and a negative characteristic.*

For example, $\log 0.005 = 3.69897$. This notation means that $\log 0.005 = -3 + 0.69897 = -2.30103$.

To transfer a negative logarithm from its natural form to the artificial form, (1) increase the absolute value of its characteristic by unity, (2) put the minus sign (a bar) over that number, (3) subtract from 9 all digits of the mantissa except the last nonzero digit; subtract the last nonzero digit from 10. The differences obtained are written in the same places of the mantissa as the digits being subtracted. Zeros at the end remain unchanged.

Example 1. Reduce $\log 0.05 = -1.30103$ to the artificial form: (1) increase by 1 the absolute value of the characteristic (which is 1); this yields 2, (2) write the characteristic in artificial form as $\bar{2}$ and place the decimal point, (3) subtract the first digit of the mantissa (which is 3) from 9; this yields 6; write the 6 in the first decimal place. The subsequent places will have the digits $9\ (=9-0)$, $8\ (=9-1)$, $9\ (=9-0)$ and $7\ (=10-3)$, or

$$-1.30103 = \bar{2}.69897$$

Example 2. Represent -0.18350 in artificial form: (1) increase 0 by 1 to get 1, (2) we have $\bar{1}$, (3) subtract from 9 the digits 1, 8, 3, and from 10 the digit 5; the zero at the end remains unchanged. This yields

$$-0.18350 = \bar{1}.81650$$

To transfer a negative logarithm from the artificial to the natural form, (1) reduce the absolute value of its characteristic by unity, (2) place the minus sign to the left of the number, (3) handle the digits of the mantissa as in the preceding case.

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* All subsequent equations are approximate to within one half unit of the last digit.

Example 3. Represent $\bar{4}.68900$ in natural form. (1) $4-1=3$, (2) we have -3, (3) subtract the digits 6, 8 of the mantissa from 9, and the digit 9 from 10; the two zeros at the end remain unchanged. This yields

$$\bar{4}.68900=-3.31100$$

130. Operations Involving Artificial Expressions of Negative Logarithms

In the case of artificial expressions of logarithms there is no need to convert them to the natural form from the start. With a little practice, the techniques described below enable one to handle artificial expressions directly just as fast as natural ones.

Addition. Mantissas are added as usual; after adding the tenths it may happen that one or more units have to be carried; if that is the case, then the carried digit is added to the positive characteristics when adding the characteristics (which may include both positive and negative numbers).

Example 1. $\bar{1}.17350+2.88694+\bar{3}.99206$.

Work:

$$\begin{array}{r} \scriptstyle 2\ 2111 \\ \bar{1}.17350 \\ +\,2.88694 \\ \bar{3}.99206 \\ \hline 0.05250 \end{array}$$

Here, adding the tenths yields $2+1+8+9=20$ (the carried digits are written at the top of the proper column). Write down 0 and carry the 2. Adding the characteristics yields $2+\bar{1}+2+\bar{3}=0$.

Example 2.

$$\begin{array}{r} \scriptstyle 1\ \ \ 11 \\ 2.7458 \\ +\,\bar{4}.3089 \\ \hline \bar{1}.0547 \end{array}$$

Adding the characteristics here we get $1+2+\bar{4}=\bar{1}$.

Subtraction. The mantissa of the subtrahend is subtracted column by column from the mantissa of the minuend both when the former is less than the latter and vice versa. In the latter case, for the tenths digit of the minuend we borrow a positive unit from the characteristic.

Example 1.

$$\begin{array}{r} \dot{\bar{2}}.\dot{1}\dot{7}\dot{4}1 \\ -\,\bar{5}.1846 \\ \hline 2.9895 \end{array}$$

In subtracting tenths we had to borrow a positive unit from the characteristic $\bar{2}$, which changed it to $\bar{3}$. Subtraction of the characteristics yields $\bar{3}-\bar{5}=2$.

Example 2.

$$\begin{array}{r} \bar{1}.\dot{2}0\dot{8}0 \\ -\,3.1916 \\ \hline \bar{4}.0164 \end{array}$$

Here we do not have to borrow from the characteristic; $\bar{1}-3=\bar{4}$.

Example 3.

$$\begin{array}{r} \dot{0}.\dot{1}\dot{2}65 \\ -\,1.9371 \\ \hline \bar{2}.1894 \end{array}$$

Here we can see that even when subtracting a positive logarithm from a positive logarithm, the result can be obtained directly in artificial form. It is advisable to do so.

In *joint addition and subtraction*, it is sometimes preferable to replace all subtractions by additions. In this case, if the subtrahend is a positive number, the corresponding negative addend is converted to artificial form. But if it is negative number specified in artificial form, it is converted to natural form and the minus sign is dropped. The resulting addends may then be called *cologarithmic addends*.

Example. $0.1535-\bar{1}.1236+\bar{1}.1686-4.3009=0.1535+$ colog. addend $\bar{1}.1236+\bar{1}.1686+$ colog. addend $4.3009=0.1535+0.8764+\bar{1}.1686+\bar{5}.6991=\bar{5}.8976$.

Work:

$$\begin{array}{r} 0.1535=0.1535 \\ -\bar{1}.1236=0.8764 \\ +\bar{1}.1686=\bar{1}.1686 \\ -4.3009=\bar{5}.6991 \\ \hline \bar{5}.8976 \end{array}$$

Multiplication. To multiply an artificial logarithm by a positive number, multiply separately first the mantissa and then the characteristic; if the multiplier is a one-digit number, then the number of positive units obtained in multiplying the mantissa is immediately added to the negative product of the multiplier into the characteristic. However, if the multiplier is multi-digit, carry the multiplication by the mantissa to the end and add the product of the multiplier by the characteristic.

Example 1.

$$\begin{array}{r} \scriptstyle 3\ 264 \\ \bar{6}.4397 \\ \times\quad 7 \\ \hline \overline{39}.0779 \end{array}$$

Example 2.

$$\begin{array}{r} \times\ \bar{1}.4397 \\ 17 \\ \hline 4397 \\ 3078 \\ \hline 7.475 \\ -\ 17 \\ \hline \overline{10}.475 \end{array}$$

Use the rules of short-cut multiplication; see Sec. 55.

If it is necessary to multiply a negative logarithm in artificial form by a negative number, it is best to convert the logarithm to natural form first.

Division. If the divisor is a negative or multi-digit positive number, it is best to convert to natural form. If the divisor is a single-digit positive number, leave the dividend in artificial form. If the characteristic is exactly divisible, then divide separately the characteristic and then the mantissa. If the characteristic is not exactly divisible, then add to it mentally a least number of negative units such that the resulting number is exactly divisible; then add mentally to the mantissa the same number of positive units.

Example. $\bar{2}.5638:6 = \bar{1}.7606$. So as to make the characteristic divisible by 6, add 4 negative units. Dividing the result, -6, by 6 yields -1. Now first add 4 positive units to the mantissa and then divide 4.5638 by 6.

131. Finding the Logarithm of a Number

The logarithms of integral powers of 10 are found without tables (Sec. 129). To find the logarithm of any other number, do as follows.

(A) *Finding the characteristic.* For number greater than unity, the characteristic is equal to the number of digits in the integral portion minus one.

Examples. $\log 35.28 = 1$ (characteristic); $\log 3.528 = 0$ (characteristic); $\log 60{,}100 = 4$ (characteristic). The numbers thus found in these examples are then followed by a decimal point and the digits of the mantissa.

For numbers less than unity, the characteristic of the artificial form of the logarithm is equal to the number of zeros preceding the significant digits of the number (including the zero in the units place).

Examples. $\log 0.00635 = \bar{3}$ (characteristic); $\log 0.1002 = \bar{1}$ (characteristic); $\log 0.06004 = 3$ (characteristic).

(B) *Finding the mantissa*. To find the mantissa of a decimal fraction (whether pure or mixed), drop the decimal point and enter the table to find the mantissa of the resultant whole number. To do this, drop all zeros (if there are any) at the end of the whole number. For example, the mantissa of the number 20.73 is equal to the mantissa of the number 2073, the mantissa of the number 6,004,800 is equal to the mantissa of the number 60,048.

When using four-place tables of logarithms, we leave only the first four digits of any whole number; if the tables are five-place we use the first five digits. The other digits are discarded because they do not affect (for all practical purposes, at any rate) the digits of the mantissa given in the table.

A four-place table gives the mantissa of a three-digit number directly; a five-place table gives the mantissa of a four-digit number directly. The mantissas of four- (five-) digit numbers are found by adding so-called mean differences, or proportional parts (see examples given below).

A four-place table of logarithms is given on pages 18-22.

Example 1. Find the logarithm of the number 45.8. We find the characteristic by inspection (without the use of the table): 1. Dropping the decimal point, we get the whole number $N = 458$. Taking the first two digits (45), move along row 45 to column 8 to find 6609. This is the mantissa. We thus have $\log 45.8 = 1.6609$.

Example 2. Find log 0.02647. We obtain the characteristic by inspection, without the table: $\bar{2}$. Dropping the decimal point we have the number 2647. Taking its first two digits (26) we move along row 26 up to column 4 (4 is the third digit of the given number) and read 4216. This is the mantissa of log 264. In the "proportional parts" of the table we find the correction corresponding to the digit 7 (the fourth digit of the given number). It is in row 26, column 7 of proportional parts and yields 11. Add this to the earlier obtained mantissa to get $4216 + 11 = 4227$. This is the mantissa of the given number. We thus have $\log 0.02647 = \bar{2}.4227$.

Work:

$$\begin{array}{r}
\log\ 0.0264 = \bar{2}.4216 \\
7 \quad\ +11 \\
\hline
\log 0.02647 = \bar{2}.4227
\end{array}$$

Note. The corrections obtained in the proportional parts data of the table are computed by means of interpolation (see Sec. 64). The use of interpolation simplifies the work of the computor. From the table we can see that the mantissa of 2640 is less than the mantissa of 2650 by $4232-4216=16$ (decimal parts). The difference of 10 between the two numbers corresponds to a difference of 16 between the mantissas. Working the proportion, we get

$$x:16=7:10=0.7, \quad x=16\cdot 0.7=11$$

Five-place table of logarithms.

Example 1. Find $\log 0.02647$. We find the characteristic by inspection: $\log 0.02647=\bar{2}$. Dropping the decimal point, we have the number 2647. Find row 264 and go along it to column 7 to find 275. These represent the last three digits of the mantissa. The first two (42) are found at the beginning of the row. The entire mantissa is 42275; $\log 0.02647=\bar{2}.42275$.

In most rows, the first two digits are not indicated. They are taken from the succeeding row (if there is an asterisk in front of the last three digits of the mantissa) or from the preceding row (if there is no asterisk).

Example 2. Find $\log 6764$. The characteristic is 3. Take row 676 of the table of logarithms and go along it to column 4 to get the last three digits of the mantissa: 020. They have an asterisk and so the first two digits (83) are taken from the next lower row 677. The entire mantissa is 83020; $\log 6764=3.83020$.

Example 3. Find $\log 6.6094$. Find the characteristic by inspection. It is 0. Drop the decimal point to get 66094. In row 660 (the first three digits of the number) we seek column 9 (the fourth digit) and find the number 014 with an asterisk. These are the last three digits of the mantissa of the number 6609. The first two (82) are found in the next row. The mantissa of $\log 6609$ is 82014. Find the correction corresponding to the last digit, 4, of the given number. In the column *PP* we find a table headed by 6 ($d=6$ is the difference between the mantissas of the numbers 6609 and 6610). We find the number 4 in the left part of this table. Opposite 4 is 2.4, which we round off to 2. This is the correction which we add to the earlier found mantissa to get $82014+2=82016$; $\log 6.6094=0.82016$.

Work:

$$\begin{array}{r}
\log 6.609 \;=0.82014\\
4 \quad + \qquad 2\\
\hline
\log 6.6094=0.82016
\end{array}$$

132. Finding a Number from a Logarithm *.

Disregard the characteristic and seek, in the table, the given mantissa or one close to it. This is used to find a certain whole number (it is found directly in the first case and with the aid of a correction, in the second; see the examples). Then examine the characteristic. If it is zero or positive, then the integral part is formed by taking one unit more than the number of units of the characteristic (zeros can be annexed at the end of the number if necessary). If the characteristic is negative, then place before the number found as many zeros as there are negative units in the characteristic; the zero on the left is set off by a decimal point. The number thus found corresponds to the given logarithm.

Four-place table (see pages 18-22).

Example 1. Find the number whose logarithm is equal to 3.4683 (that is, the number $10^{3.4683}$). First in the table seek the mantissa 4683 or one close to it. Run down one of the columns, say column 0, and seek a number whose first two digits are 46 or a number close to 46. We find such a number (4624) in row 29. Thereabouts look for the mantissa 4683; we find it in row 29, column 4. Hence, the number with mantissa 4683 is 294. Since the characteristic 3 is positive, we take $3+1=4$ digits for the integral part. And so we annex a zero at the end of 294. This yields $3.4683=\log 2940$.

Example 2. Find the number whose logarithm is $\bar{3}.3916$. Proceeding as in the previous example, we do not find the number 3916 among the mantissas, but we find a close number, 3909, at the intersection of row 24 and column 6. Thus, the number 246 corresponds to the mantissa 3909, which yields the first three significant digits of the desired number. The fourth digit is found by computing the correction. The given mantissa 3916 exceeds the tabular value 3909 by 7. We seek this digit in the same row 24 in "Proportional Parts": It is found in column 4. The digit 4 is thus the fourth significant digit of the desired number; the number 2464 corresponds to the mantissa 3916. Now examine the characteristic. Since it is negative and contains three units, we put three zeros in front of the number we found and set off one decimal place. We thus have $\bar{3}.3916=\log 0.002464$.

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* When looking up a number on the basis of its four-place logarithm, it is best to use a table of antilogarithms (see Sec. 133). It is not advisable for five-place work to double the volume of a logarithmic table by adjoining a table of antilogarithms.

Work:

$$\log x = \bar{3}.3916$$

$$\begin{array}{ll} 3909 & \log 246 \\ \underline{+7} & \underline{4}; \quad x = 0.002464 \\ 3916 & \log 2464 \end{array}$$

Note 1. Bear in mind firmly that when seeking a number on the basis of a logarithm the proportional parts correction is annexed to it and not added to the last digit.

Note 2. Do not forget that the correction is to be sought in the same row as the number that is our approximation to the mantissa. If this row does not have the correction of the mantissa we need, then take the next closest correction.

Five-place table of logarithms.

Example 1. Find the number whose logarithm is $\bar{2}.43377$. Turn the pages noting the first two digits of the mantissas (the numbers increase). Find 43 and then in the vicinity look for the last three digits, or 377. These digits are located at the intersection of row 271 and column 5. The number with the mantissa 43377 is thus 2715. Taking into account the characteristic ($\bar{2}$), we have $\bar{2}.43377 = \log 0.02715$.

Note. In most cases, the last three digits of a mantissa are found in the same row as the first two digits, or in one of the rows below; it is, however, possible that the last three digits will be the nearest row above, in which case they are preceded by an asterisk.

Example 2. Find a number whose logarithm is 0.14185. Proceed in the same manner as in the above example; we do not find 14185 among the mantissas but we do find 14176, which is close. The last three digits of the mantissa (176) lie above the first two and so we find an asterisk in front. The mantissa 14176, which stands at the intersection of row 138 and column 6, is associated with the number 1386, which yields the first four digits of the desired number. The fifth digit is computed by interpolation. The mantissa at hand exceeds the tabular value by $185-176=9$. Now the difference between the two closest tabular mantissas is $208-176=32$.

In the column *PP* we find a small table headed 32. In it, to the right, we look for a number close to 9 and find 9.6. Opposite this number we find 3. This digit is the fifth significant digit of the desired number. The number having the mantissa 14185 is 13863. Taking into account the characteristic, we get $0.14185 = \log 1.3863$.

Work:

$$\begin{array}{rrl} \log x = 0.14185 & & \\ 14176 & 1386 & \\ +9 & 3; & \quad x = 1.3863 \\ \hline 14185 & 13863 & \end{array}$$

Note. When finding a number from a logarithm, the proportional parts data are annexed to the number and are not added to the last digit.

133. Tables of antilogarithms

The so-called table of antilogarithms (see pages 23-27) is the same as a logarithmic table only the data are arranged differently so as to simplify finding a number from a given logarithm. Only mantissas (denoted by m) are given in the table (in boldface type). If the mantissa has three decimal places, the table gives a whole number directly; if the mantissa has four decimal places, the number is found with the aid of proportional parts data (see examples). The given characteristic is then inspected. If it is zero or positive, then the integral part has one more digit than the number of units in the characteristic (any number of required zeros may be annexed at the end of the number). If the characteristic is negative, then the number is preceded by as many zeros as there are units in the characteristic. The zero at the extreme left is set off by a decimal point. The number thus found corresponds to the given logarithm.

Example 1. Find a number whose logarithm is equal to 2.732 (that is, the number $10^{2.732}$). Disregard the characteristic and take the first two digits of the mantissa (73). Go along row 73 up to column 2 to find the number 5395. Since the characteristic 2 is positive, the integral part has $2+1=3$ digits. Our answer is $10^{2.732}=539.5$.

Example 2. Given $\log x=\bar{3}.2758$. Find x. Disregard the characteristic and find the number in row 27 and column 5. It is 1884. Find the digit in the proportional parts data corresponding to the digit 8. It is 3. Add it to the number that was found: $1884+3=1887$. Inspect the characteristic. Since it is negative and contains three units, we put three zeros in front of the number 1887 and set off one decimal place from the left. We have

$$x=0.001887 \text{ or } \log 0.001887=\bar{3}.2758$$

Work:

$$\begin{array}{lr} \log x = 3.2758 & \\ 275 & 1884 \\ \underline{\quad 8+} & \underline{\quad 3} \\ 2758 & 1887 \end{array}$$

$$x = 0.001887$$

Example 3. $\log x = 0.0817$. Find x.

$$\begin{array}{lr} 081 & 1205 \\ \underline{\quad 7+} & \underline{\quad 2} \\ 0817 & 1207 \end{array}$$

$$x = 1.207$$

Note. When finding a number from a logarithm with the aid of antilog tables, the proportional parts data are always added to the last digit and not annexed.

134. Logarithmic Computations (Worked Examples)

Example 1. Compute $u = \frac{ab}{\sqrt{a^2 - b^2}}$ where $a = 4.352$, $b = 1.800$.

(1) Taking logs,

$$\log u = \log \frac{ab}{\sqrt{a^2 - b^2}} = \log \frac{ab}{\sqrt{(a+b)\ (a-b)}}$$

$$= \log a + \log b - \frac{1}{2} [\log (a+b) + \log (a-b)]$$

(2) We find $a+b$ and $a-b$:

$$+\begin{array}{r} a = 4.352 \\ \underline{b = 1.800} \\ a + b = 6.152 \end{array} \qquad -\begin{array}{r} a = 4.352 \\ \underline{b = 1.800} \\ a - b = 2.552 \end{array}$$

(3) First compute $\log a + \log b$, then $\frac{1}{2} [\log (a+b) + \log (a-b)]$:

$$\begin{array}{r} \log a = \log 4.352 = 0.6387 \\ \underline{\log b = \log 1.800 = 0.2553} \\ \log a + \log b = 0.8940 \end{array}$$

$$\begin{array}{r} \log (a+b) = \log 6.152 = 0.7890 \\ \underline{\log (a-b) = \log 2.552 = 0.4068} \\ \log (a+b) + \log (a-b) = 1.1958 \\ \frac{1}{2} [\log (a+b) + \log (a-b)] = 0.5979 \end{array}$$

(4) Find $\log u$ and then u:

$$\begin{array}{r} -\begin{array}{r} 0.8940 \\ 0.5979 \end{array} \\ \hline \log u = 0.2961, \quad u = 1.977 \end{array}$$

Example 2. Evaluate $P = pe^{-\frac{k}{p}h}$ where $p = 10.33$, $k = 0.00129$, $h = 1000$, and e is the base of natural logarithms ($e \approx 2.7183$).

$$(1)\quad \log P = \log p - \frac{k}{p} h \log e = \log p - \frac{k}{p} hM,$$

where $M = \log e \approx 0.4343$ (the modulus of common logarithms with respect to natural logarithms; see Sec. 128).

(2) Find $\log p$:

$$\log p = \log 10.33 = 1.0141$$

(3) Take the logarithms of the expression $\frac{k}{p} hM$:

$$\log \frac{k}{p} hM = \log k + \log h + \log M - \log p$$

(4) Evaluate this logarithmic expression:

$$\begin{array}{rl} \log k = \log 0.00129 & = \bar{3}.1106 \\ \log h = \log 1000 & = 3.0000 \\ \log M = \log 0.4343 & = \bar{1}.6378 \\ \text{colog } p = \text{colog } 10.33 & = \bar{2}.9859 \\ \hline \log \frac{k}{p} hM & = \bar{2}.7343 \end{array}$$

Whence $\frac{k}{p} hM = 0.05424$.

(5) Evaluate $\log P$ (see Item 1) and then P:

$$\begin{array}{rl} \log p & = 1.0141 \\ -\frac{k}{p} hM & = 0.0542 \\ \hline \log P & = 0.9599, \text{ whence } P = 9.118 \end{array}$$

135. Combinatorics (Permutations and Combinations)

There are two basic ways of arranging items chosen from some set of distinct objects (elements). They are termed *permutations* and *combinations*.

1. Permutations. Take m distinct objects (elements) a_1, a_2, $\ldots$, a_m. Rearrange these elements in all possible ways keeping their number constant and only changing their order. Each such arrangement (including the original one) is called a *permutation*. The total number of permutations of m elements is denoted by P_m. This number is equal to the product of all integers from 1 (or, what is the same thing, from 2) to m inclusive:

$$P_m = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (m-1) \cdot m = m! \tag{1}$$

The symbol $m!$ (read "*m factorial*") denotes the product $1 \cdot 2 \cdot 3 \cdot \ldots \cdot (m-1) \cdot m$.

Example 1. Find the number of permutations of three elements a, b, c. We have $P_3 = 1 \cdot 2 \cdot 3 = 6$. Indeed, we have 6 permutations:

(1) *abc*, (2) *acb*, (3) *bac*, (4) *bca*, (5) *cab*, (6) *cba*.

Example 2. In how many ways can five posts be assigned to five persons elected to the administrative board of a sports club? Write a list of the posts and opposite each write a name. This will be one permutation. The total number of such permutations is $P_5 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

Note. When $m = 1$ in the expression $1 \cdot 2 \cdot 3 \cdot \ldots \cdot m$, there is only one number, 1. It is therefore agreed (by definition) that $1! = 1$. For $m = 0$ the expression $1 \cdot 2 \cdot 3 \cdot \ldots \cdot m$ becomes meaningless, and so it is defined that $0! = 1$. See Item 2 below for the justification of this convention.

Let us now take this set of m elements and make up groups of n elements in each, arranging the n elements in different order. The resulting arrangements are termed *permutations* of m elements taken n at a time. The total number of permutations of m elements n at a time is denoted by P_m^n (or ${}_nP_m$). This number is equal to the product of n successive integers of which the largest is m:

$$P_m^n = m(m-1)(m-2) \ldots [m-(n-1)] \tag{2}$$

Example 3. Find the number of permutations of four elements *abcd* taken two at a time. We have $P_4^2 = 4 \cdot 3 = 12$. These permutations are the following:

ab, *ba*, *ac*, *ca*, *ad*, *da*, *bc*, *cb*, *bd*, *db*, *cd*, *dc*

Example 4. A meeting elects 8 persons to its supreme body, which then chooses a chairman, a secretary and a treasurer. In how many ways can these posts be assigned?

The desired number is the number of permutations of 8 elements taken 3 at a time, or $P_8^3 = 8 \cdot 7 \cdot 6 = 336$.

2. **Combinations.** Take m distinct elements and make up groups of n elements in each but disregard the order of the elements in each group. We then have *combinations* of m elements taken n at a time.

The total number of distinct combinations is denoted by C_m^n. This number (which is integral of course) can be represented by the formula * (see Item 1)

$$C_m^n = \frac{P_m}{P_n P_{m-n}} = \frac{m!}{n!\,(m-n)!} \tag{3}$$

By definition we assume $C_m^0 = 1$ [this value is obtained from (3)].

The expression $\frac{m!}{n!\,(m-n)!}$ is often abbreviated to $\binom{m}{n}$.

It is clear that $\binom{m}{n} = \binom{m}{m-n}$, that is, $C_m^n = C_m^{m-n}$.

In computations it is often more convenient to make use of other expressions for the number of combinations, namely

$$C_m^n = \frac{P_m^n}{P_n} = \frac{m\,(m-1)\ldots[m-(n-1)]}{1 \cdot 2 \cdot \ldots \cdot n}$$

or

$$C_m^n = \frac{P_m^{m-n}}{P_{m-n}} = \frac{m\,(m-1)\ldots(n+1)}{1 \cdot 2 \ldots (m-n)}$$

Example 5. Find all the combinations of five elements *abcde* taken three at a time. We have $C_5^3 = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 10$. These ten combinations are

abc, *abd*, *abe*, *acd*, *ace*, *ade*, *bcd*, *bce*, *bde*, *cde*

Example 6. There are eight candidates for three counters. In how many ways can the assignments be made? Since the duties of each counter are the same, we have combinations and not permutations, as in Example 4 of Item 1. The sought-for number is

$$C_8^3 = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$$

Combinatorial mathematics deals with many more types of arrangements than those described above. One of the most important types is *permutations with repetitions of the elements*. These are defined as follows. Take m elements, of which m_1 are identical elements of type one, m_2 are identical elements of a second kind, etc. Permute them in all possible ways. We then get *permutations with repetitions*. The number of distinct permutations with repetitions is

---

* There is only one combination containing all m elements and so $C_m^m = 1$. Formula (3) gives this value only if we agree that $0! = 1$.

$$\frac{P_m}{P_{m_1}P_{m_2}\ldots P_{m_k}} \quad \text{or} \quad \frac{m!}{m_1!\,m_2!\ldots m_k!}$$

$(m_1+m_2+\ldots+m_k=m$, and k is the number of kinds).

Example 7. Find the number of distinct permutations of the letters *aaabbcc* with repetitions of the elements. Interchanging the first two letters does not result in a new arrangement. The same occurs when we interchange the fourth and fifth letters, and in other cases. But the arrangements *abaabcc*, *caaabcb* and certain others are distinctly new ones. In this example, $m_1=3$, $m_2=2$, $m_3=2$; $m=m_1+m_2+m_3=7$. The number of distinct permutations is

$$\frac{7!}{3!\,2!\,2!}=\frac{2\cdot3\cdot4\cdot5\cdot6\cdot7}{2\cdot3\cdot2\cdot2}=210$$

Example 8. Find the number of distinct permutations made up of the signs $++++---$. Here, $m_1=4$, $m_2=3$; $m=m_1+m_2=7$. The desired number is $\frac{7!}{4!\,3!}=35$. From this example it is easy to see that the number of permutations of m elements, among which are repeated m_1 elements of the first kind and m_2 elements of the second kind, is equal to the number of combinations of m elements taken m_1 at a time, or to the number of combinations of m elements taken m_2 at a time. Indeed, each permutation is associated with one and only one selection of positions with the $+$ sign. Thus, in the permutation $++--+-+$ the $+$ signs occupy the 1, 2, 5, 7 positions so that the corresponding combination is 1, 2, 5, 7. Hence, there are as many permutations as there are distinct combinations of seven numbers taken four at a time.

136. The Binomial Theorem

Newton's binomial theorem states a formula expressing $(a+b)^n$ in the form of a polynomial for positive integral n*.

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* The term "Newton's binomial theorem" is a misnomer, firstly because $(a+b)^n$ is not a binomial, secondly, the expansion of $(a+b)^n$ for positive integral n was known before Newton's time. But to Newton goes the credit for the bold and extremely fruitful idea of extending the expansion to the case of negative and fractional n.

The binomial formula for positive integral n is

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \dots + \binom{n}{n-1} ab^{n-1} + b^n \qquad (1)$$

or, what is the same thing (see p. 242),

$$(a+b)^n = a^n + \frac{n!}{1!\,(n-1)!} a^{n-1} b + \frac{n!}{2!\,(n-2)!} a^{n-2} b^2 + \dots \qquad (2)$$

Accordingly, it is assumed that $\binom{n}{0} = \binom{n}{n} = 1$ and also $0! = 1$ (see note on page 242). With this convention, the first and last terms of the expansion are of the same form as the other terms.

For computational purposes it is more convenient to use the formula

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} a^{n-3}b^3 + \dots + b^n \qquad (3)$$

Example 1. $(a+b)^3 = a^3 + 3a^2b + \frac{3\cdot 2}{1\cdot 2} ab^2 + b^3 = a^3 + 3a^2b + 3ab^2 + b^3.$

Example 2. $(1+x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6.$

The numbers $1, n, \frac{n(n-1)}{1\cdot 2}, \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$, etc. are termed *binomial coefficients*. They may be obtained in the following manner using only addition. In the top row (see accompanying array) write two units. All other rows begin and end with unity. The intermediate numbers are found by adding adjacent numbers of the row above. Thus, the number 2 in the second row is found by adding the two units of the first row; the third row is obtained from the second this way: $1+2=3$, $2+1=3$; the fourth row from the third as follows: $1+3=4$, $3+3=6$, $3+1=4$, and so forth. The numbers in one row are the binomial coefficients of an appropriate power. This array is called *Pascal's triangle* (or the arithmetic triangle).

$$\begin{array}{ccccccccccccc}
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

The Binomial Theorem for Fractional and Negative Exponents

Suppose we have the expression $(a+b)^n$ where n is a fractional or negative number. Let $|a|>|b|$. Represent $(a+b)^n$ as $a^n(1+x)^n$. The quantity $x=\frac{b}{a}$; its absolute value is less than unity. The expression $(1+x)^n$ may be computed to any desired degree of accuracy by formula (3).

Example 1. $\frac{1}{1+x}=(1+x)^{-1}$. Here $n=-1$.

Since $\frac{n(n-1)}{1\cdot 2}=\frac{(-1)\cdot(-2)}{1\cdot 2}=1$,

$$\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}=\frac{(-1)\cdot(-2)\cdot(-3)}{1\cdot 2\cdot 3}=-1$$

and so on, we have $(1+x)^{-1}=1-x+x^2-x^3+x^4-\dots$.

The number of terms in the right-hand member is infinitely great, but when $|x|<1$, the sum of the terms, as their number increases without bound, tends to the limit $\frac{1}{1+x}$ (the expression in the right member, if $|x|<1$, is an infinitely decreasing geometric series).

Example 2. Compute $\sqrt{1.06}$ to five decimal places.

Represent $\sqrt{1.06}$ as $(1+0.06)^{\frac{1}{2}}$ and apply formula (3):

$$(1+0.06)^{\frac{1}{2}}=1+\frac{1}{2}\cdot 0.06+\frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)}{1\cdot 2}\cdot 0.06^2$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1\cdot 2\cdot 3}\cdot 0.06^3+\dots=1+0.03-0.00045$$
$$+0.0000135-\dots$$

Subsequent terms do not affect the first five decimal places and so, summing the four terms written out, we have

$$\sqrt{1.06}\approx 1.02956$$

Example 3. Find five significant figures of the number $\sqrt[3]{130}$.

The cube nearest to 130 is $125=5^3$. Represent $\sqrt[3]{130}$ as $(125+5)^{\frac{1}{3}}=125^{\frac{1}{3}}(1+0.04)^{\frac{1}{3}}=5(1+0.04)^{\frac{1}{3}}$. Carry the answer to seven places (taking note of the fact that the error is accumulative in addition and is then increased fivefold):

$$(1+0.04)^{\frac{1}{3}}=1+\frac{1}{3}\cdot 0.04+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{1\cdot 2}\cdot 0.04^2$$

$$+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{1\cdot 2\cdot 3}\cdot 0.04^3+\dots$$

$$=1+0.0133333-0.0001778+0.0000040-\dots=1.0131595$$

The discarded terms do not affect the seventh digit and we find $5\cdot 1.0131595=5.0657975$. Accurate to the fifth decimal place, we have $\sqrt[3]{130}=5.06580$. A more precise computation (with regard for the next term) yields 5.0657970, all decimals correct.

Using this device we can extract the root of any degree of any number rapidly and to any desired degree of accuracy.

Generalized Binomial Formula

$$(a_1+a_2+a_3+\dots+a_k)^n=\sum\frac{n!}{n_1!n_2!\dots n_k!}a_1^{n_1}a_2^{n_2}\dots a_k^{n_k}$$

where n is a positive integer.

The symbol $\sum$ signifies that we must take the sum of all terms of the form

$$\frac{n!}{n_1!n_2!\dots n_k!}a_1^{n_1}a_2^{n_2}\dots a_k^{n_k}$$

where n is the given exponent and $n_1, n_2, \dots, n_k$ are arbitrary integers or zeros whose sum is equal to n. The number $0!$ is taken equal to 1.

Example

$$(a+b+c+d)^3=\sum\frac{3}{n_1!\,n_2!\,n_3!\,n_4!}a^{n_1}\,b^{n_2}\,c^{n_3}\,d^{n_4}$$

the number $n=3$ may be given as the sum of $k=4$ integral sumands thus:

$$3=3+0+0+0,$$
$$3=2+1+0+0,$$
$$3=1+1+1+0$$

Accordingly, we have

$$(a+b+c+d)^3=\frac{3!}{3!\,0!\,0!\,0!}(a^3b^0c^0d^0+a^0b^3c^0d^0$$
$$+a^0b^0c^3d^0+a^0b^0c^0d^3)+\frac{3!}{2!\,1!\,0!\,0!}(a^2bc^0d^0+ab^2c^0d^0$$
$$+a^2b^0cd^0+ab^0c^2d^0+\dots)$$
$$+\frac{3!}{1!\,1!\,1!\,0!}(abcd^0+abc^0d+ab^0cd+a^0bcd)$$
$$=a^3+b^3+c^3+d^3+3(a^2b+ab^2+a^2c+ac^2+a^2d+ad^2$$
$$+b^2c+bc^2+b^2d+bd^2+c^2d+cd^2)+6(abc+abd+acd+bcd)$$

Properties of Binomial Coefficients

1. The coefficients of terms the same distance from the ends of the expansion are the same.

For example, in the expansion

$$(a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$$

the coefficients of the second and second to the last terms are the same, 6; the coefficients of the third term from the beginning and the third from the end are the same, 15.

2. The sum of the coefficients of the expansion of $(a+b)^n$ is equal to 2^n. For example, in the expansion above we have

$$1+6+15+20+15+6+1=64=2^6$$

3. The sum of the coefficients of terms in odd positions is equal to the sum of the coefficients of terms in even positions. Each one is 2^{n-1}; for example, in the expansion $(a+b)^6$ the sum of the coefficients of the 1st, 3rd, 5th and 7th terms is equal to the sum of the coefficients of the 2nd, 4th and 6th terms:

$$1+15+15+1=6+20+6=32=2^5$$

GEOMETRY

A. PLANE GEOMETRY

137. Geometric Constructions

1. *To draw, through a given point C, a straight line parallel to a given straight line AB.*

Open a pair of compasses and draw an arbitrary circle (Fig. 22), centre C, such that it intersects AB. Using the same opening of the compasses lay off on AB from one of the points of intersection, say M, a line-segment MN (in either direction) and then describe the arc ab from N. Joint C to the point P of intersection of the arc ab with the circle. PC is the desired straight line.

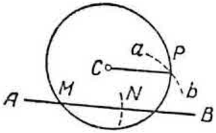

Fig. 22.

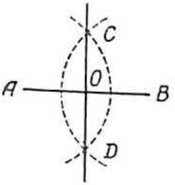

Fig. 23.

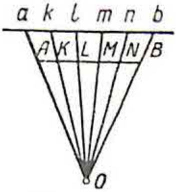

Fig. 24.

2. *To bisect a given line-segment AB.*

Using a pair of compasses with an arbitrary opening (but greater than $^{1}/_{2}\,AB$) describe two arcs from the endpoints of the line AB. Join their points of intersection (C and D) with a straight line. The intersection point O of the straight lines AB and CD is the midpoint of the line AB.

3. *To divide a given line AB into a given number of equal parts.*

In Fig. 24 draw a straight line ab parallel to AB; on it lay off as many equal segments of arbitrary length as needed, say $ak = kl = lm = mn = nb$. Draw the straight lines Aa, Bb to intersect in the point O. Draw straight lines Ok, Ol, Om, On. These lines will intersect AB at the points K, L, M, N, which divide AB into the required number (in our case, 5) of equal parts.

4. *To divide a given line-segment into parts proportional to given quantities.*

This problem is solved in the same way as Problem 3, but on *ab* we lay off segments proportional to given quantities.

5. *To draw a straight line perpendicular to a given straight line MN from a given point A.*

From an arbitrary point *O* without the given straight line (Fig. 25) draw a circle of radius *OA*. Draw a diameter

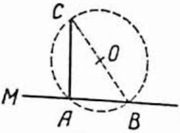

Fig. 25.

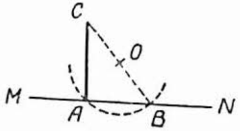

Fig. 26.

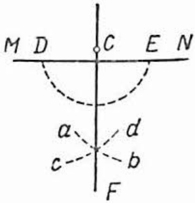

Fig. 27.

BC through the second point *B* of intersection of the circle with the line *MN*. Join *A* and the endpoint of the diameter, *C*; *CA* is the desired perpendicular.

This mode of construction is particularly useful when the point *A* lies close to the edge of the paper. The method of solving the next problem (first case) has the same advantage.

6. *To drop a perpendicular from a given point C to a straight line MN.*

From point *C* draw an arbitrary inclined line *CB* (Fig. 26). Find its midpoint *O* (see Problem 2) and from it describe a circle of radius *OB*. The circle also intersects *MN* at point *A*. Join *A* and *C* to get the desired perpendicular.

When *C* lies close to *MN*, this method of construction may result in a considerable error. The following is a preferable alternative construction. From point *C* as centre (Fig. 27), draw an arc *DE* cutting *MN* at points *D* and *E*. From *D* and *E* as centres, draw two arcs *cd*, *ab* of the same radius; they intersect at *F*. Draw *FC* to obtain the desired perpendicular.

7. *For a given vertex K and ray KM, to construct an angle equal to a given angle ABC.*

From the vertex *B* (Fig. 28) describe an arc *PQ* of arbitrary radius. Using the same opening of the compasses, describe from the centre K an arc pq. From point p describe an arc $\alpha\beta$ of radius equal to PQ. Join to K the intersection point q of arcs pq and $\alpha\beta$. The angle qKM is the required angle.

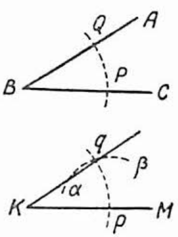

Fig. 28.

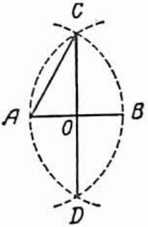

Fig. 29.

8. *To construct angles of* 60° *and* 30°.

From endpoints A and B (Fig. 29) of an arbitrary line AB describe two arcs of radius AB. Join their intersection points C and D with a straight line which cuts AB at its midpoint O. Join A and C with a straight line. $\angle CAO = 60°$, $\angle ACO = 30°$.

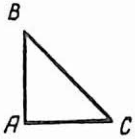

Fig. 30.

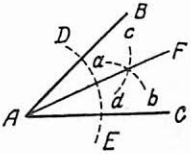

Fig. 31.

9. *To construct an angle of* 45°.

Lay off on the sides of a right angle BAC (Fig. 30) equal segments AB and AC and join their endpoints with a straight line CB. The line BC forms 45° angles with AC and AB.

10. *To bisect a given angle* BAC.

From vertex A (Fig. 31) draw an arc DE of arbitrary radius. From points D and E (where DE cuts arms AB and AC) draw the arcs ab, cd with arbitrary equal radii (it is most convenient to use the original opening of the compasses). Join their point of intersection to A. The resulting straight line AF bisects the angle BAC.

11. *To trisect a given angle BAC.*

It is impossible to make this construction with straightedge and compasses alone. Using compasses and a graduated ruler (say with centimetre divisions) the construction can be carried out as follows (Fig. 32): from point A describe a circle of arbitrary radius AC. Produce AC beyond A. Place the ruler so that it passes through B, then revolve it about B until ED (the line-segment between the circle and the straight line AK) is equal to the radius AC. The angle EDF is one third of the angle BAC.

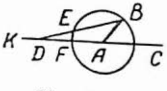

Fig. 32.

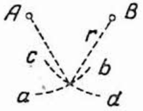

Fig. 33.

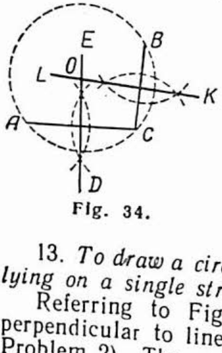

Fig. 34.

12. *Through two given points A and B to draw a circle of given radius r.*

From points A and B (Fig. 33) describe arcs ab and cd of radius r. Their point of intersection is the centre of the desired circle.

13. *To draw a circle through three given points A, B, C not lying on a single straight line.*

Referring to Fig. 34, draw straight lines ED and KL perpendicular to lines AC and BC at their midpoints (see Problem 2). The point O of intersection of these perpendiculars is the centre of the desired circle.

14. *To find the centre of a given arc of a circle.*

Take three points on the arc spaced as far apart as possible. Then proceed as in Problem 13.

15. *To bisect a given arc of a circle.*

Join the ends of the arc by a chord. Draw a perpendicular through the midpoint of the chord (see Problem 2). It bisects the arc (and the chord).

16 *To find the locus from which a given line AB is seen at a given angle α.*

The desired locus (Fig. 35) is in the form of two arcs of equal circles with endpoints at A and B. (The points A and B do not belong to the locus). The centres of these arcs are found as follows: draw perpendiculars AD and BK to the endpoints of AB (see Problem 5). Construct an angle $KBL=\alpha$. We find point C at the intersection of BL and AD. The midpoint O of line BC is the centre of one of the sought-for arcs. The other arc is constructed similarly.

17. *To draw through a given point A a tangent to a given circle.*

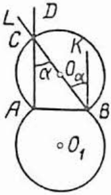

Fig. 35

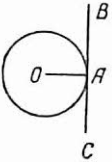

Fig. 36.

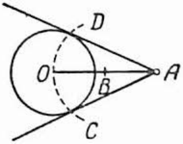

Fig. 37.

If A lies on the circle (Fig. 36), construct BAC perpendicular to radius OA (see Problem 5); CB is the required tangent line.

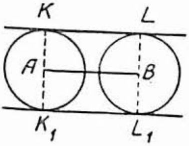

Fig. 38.

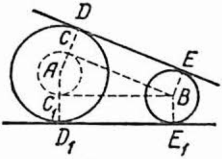

Fig. 39.

If A lies without the circle (Fig. 37), bisect AO (see Problem 2) and from the midpoint B draw arc CD of radius BO. Join D and C to A with straight lines. The straight lines AD and AC are the required tangent lines.

18. *To construct an exterior common tangent to two given circles.*

(a) If the radii of the given circles are equal, the problem always has two solutions (Fig. 38). Through centres A and B draw diameters KK_1 and LL_1 perpendicular to the line AB of centres. Drawing KL and K_1L_1 we get the required solutions.

(b) Let the radii of the circles be unequal: $R > r$; from the centre of the larger circle draw a circle of radius $AC=R-r$ (Fig. 39). Draw to this circle from the centre B of the smaller circle the tangent line BC (Problem 17). Join centre A to the point of contact C. Produce it to meet the larger circle at D. Draw BE perpendicular to BC to meet the smaller circle at E. Join D and E. The straight line DE is the desired tangent. The problem admits two solutions (DE and D_1E_1) if the smaller circle does not lie wholly within the larger one. If the smaller circle lies wholly within the larger one (Fig. 40), the problem does not have a solution. In the intermediate case when the circles are tangent internally (Fig. 41), the problem has one solution: through the point M of internal contact draw $KL \perp AM$.

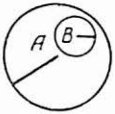

Fig. 40.

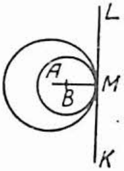

Fig. 41.

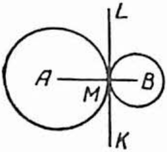

Fig. 42.

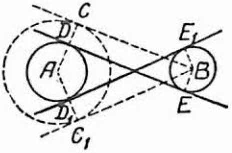

Fig. 43.

19. *To construct an interior common tangent to two given circles.*

The problem has no solution if one of the circles lies within the other and also if the circles intersect. In the case of external contact (Fig. 42) the problem has one solution: draw $KL \perp AB$ through M.

In all other cases we have two solutions (DE and D_1E_1, Fig. 43). From centre A draw a circle of radius equal to the sum of the radii of the given circles. From centre B draw the tangent BC to the constructed circle (Problem 17). Join the point of contact C and the centre A; the straight line AC cuts the circle (A) at point D. From B draw a radius $BE \perp BC$. Join its extremity E to D; ED is the desired tangent. The other tangent, E_1D_1, is constructed analogously.

20. *To circumscribe a circle about a given triangle ABC.* Draw a circle through vertices A, B, C (see Problem 13).

21. *To inscribe a circle in a given triangle ABC.*

In Fig. 44 bisect two angles of the triangle, say A and C (see Problem 10). From the point O of intersection of the bisectors draw $OD \perp AC$ (see Problem 6). Using radius OD describe the required circle.

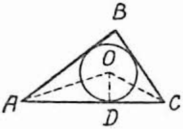

Fig. 44.

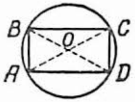

Fig. 45.

22. *To circumscribe a circle about a given rectangle (or square) ABCD.*

Draw diagonals BD and AC (Fig. 45). From the point O of their intersection draw a circle of radius OA.

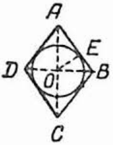

Fig. 46.

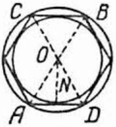

Fig. 47.

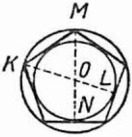

Fig. 48.

It is impossible to circumscribe a circle about an oblique-angled parallelogram.

23. *To inscribe a circle in a rhombus (or square) ABCD.*

From the point O of intersection of the diagonals draw $OE \perp AB$ (Fig. 46). The circle with centre O and radius OE is the required circle.

It is not possible to inscribe a circle in a nonequilateral parallelogram.

24. *To circumscribe a circle about a given regular polygon.*

If the number of sides is even (Fig. 47), join any two pairs of opposite vertices by straight lines AB and CD. From the point O of their intersection describe a circle of radius OA.

If the number of sides is odd (Fig. 48), drop perpendiculars KL and MN from vertices K and M onto opposite sides. From the intersection point O describe a circle of radius OK.

25. *To inscribe a circle in a given regular polygon.*

The centre of the circle is found as in Problem 24. From the centre drop a perpendicular ON on one of the sides (see Fig. 47). Describe the circle using radius ON (or OL, Fig. 48).

26. *To construct a triangle, given three sides a, b, and c.*

Let the longest be a. If $a < b+c$, then the desired triangle can be constructed as follows: lay off $BC = a$ (Fig. 49). From its extremities B and C describe arcs mn and pq of radii c and b. Join the point A of intersection of the arcs to B and C.

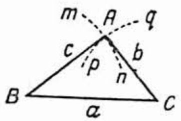

Fig. 49.

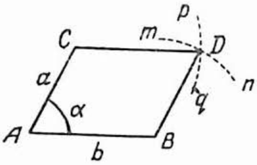

Fig. 50.

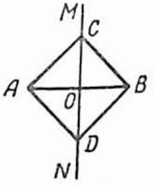

Fig. 51.

If $a > b+c$, then the problem has no solution. In the intermediate case $a = b+c$, only a degenerate triangle meets the conditions: its three vertices lie on one straight line.

27. *To construct a parallelogram, given sides a* and b and one angle α.

Construct $\angle A = \alpha$ (see Problem 7); on its sides lay off $AC = a$, $AB = b$ (Fig. 50). Draw from B an arc mn of radius a and from C an arc pq of radius b. Join the point D of intersection of these arcs to C and B.

28. *To construct a rectangle, given the base and the altitude.*

Proceed as in Problem 27; construct the right angle α as in Problem 5.

29. *To construct a square, given a side.*

Proceed as in Problems 27 and 28.

30. *To construct a square, given its diagonal AB.*

Through the midpoint of AB (Fig. 51), draw a perpendicular MN to AB (see Problem 2). From O, its point of intersection with AB, lay off on MN lengths OC and OD equal to OA. $ACBD$ is the required square.

31. *To inscribe a square in a given circle.*

Draw two mutually perpendicular diameters AB and CD (Fig. 52). $ACBD$ is the required square.

32. *To circumscribe a square about a given circle.*

Draw two mutually perpendicular diameters AB and CD (Fig. 53). From their extremities as centres, describe four semicircles of radii OA. The points of their intersection, F, G, H, and E, are the vertices of the required square.

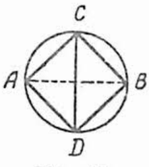

Fig. 52.

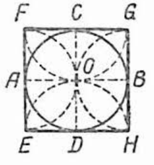

Fig. 53.

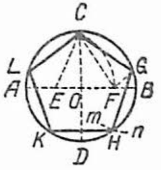

Fig. 54.

33. *To inscribe a regular pentagon in a given circle.*

Draw two mutually perpendicular diameters AB and CD (Fig. 54). Bisect the radius AO to get point E. From E draw an arc CF of radius EC, cutting the diameter AB at F. From C draw an arc FG of radius CF cutting the given circle at G; CG $(=CF)$ is one side of the required figure. With the same radius, draw arc mn from centre G to get one more vertex, H, of the desired figure, etc.

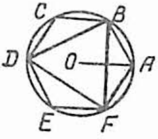

Fig.55.

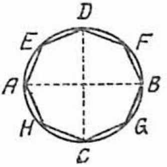

Fig. 56.

34. *To inscribe a regular hexagon and a triangle in a circle.*

Opening the compasses to the radius of the circle, strike arcs on the circumference at points A, B, C, D, E, F (Fig. 55). Join A, B, C, D, E, F in succession to obtain a regular hexagon. Joining alternate points, we get a regular (equilateral) triangle.

35. *To inscribe a regular octagon in a given circle.*

Draw two mutually perpendicular diameters AB and CD (Fig. 56). Bisect arcs AD, DB, BC, CA by points E, F, G, H (Problem 15). Join in succession the eight points thus obtained.

36. *To inscribe a regular decagon in a given circle.*

Construct a point F (see Fig. 54) as in Problem 33. OF is a side of the required figure. Opening the compasses to the length OF, strike 10 consecutive arcs on the circumference. These are the vertices of the desired figure.

Regular polygons inscribed in a circle and having 7 and 9 sides are not constructible exactly by means of straightedge and compasses.

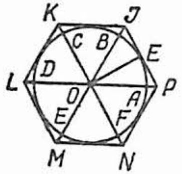

Fig. 57.

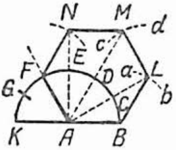

Fig. 58.

37. *To circumscribe about a circle a regular triangle, pentagon, hexagon, octagon, and decagon.*

Mark on the circumference of a circle (Fig. 57) the vertices A, B, ..., F of a regular inscribed polygon with the same number of sides (see Problems 33 to 36). Draw radii OA, OB, ..., OF and produce them. Bisect the arc AB by the point E (see Problem 15). Through E draw $JP \perp OE$. The length JP between the extensions of adjacent radii is a side of the required figure. On the extensions of the other radii lay off lengths OK, OL, ..., ON equal to OP. Join the points J, K, L, ..., N, P in succession. The polygon $JKLM \ldots NP$ is the required one.

38. *To construct a regular n-gon, given a side a.*

On a length BK (Fig. 58), equal to $2a$, as diameter, construct a semicircle. Divide the semicircle into n equal parts by the points C, D, E, F, G (these are the vertices of a regular inscribed n-gon; $n=6$ in our figure). Connect centre A with all the points obtained, except the last two (K and G). From B draw an arc ab of radius AB, marking point L on the ray AC. Using the same radius, from point L draw an arc cd, marking point M on the ray AD, etc. Join the points B, L, M, N and so on in succession. The polygon $ABLMNF$ is the required one.

The problem is not always solvable by straightedge and compasses; for instance, for $n=7$ and $n=9$ the figure is not constructible since it is impossible, using straightedge and compasses, to divide a semicircle into 7 or 9 equal parts.

138. The Subject of Geometry

Geometry (see Sec. 139 on the origin of the word) studies the spatial properties of objects disregarding all other features. For instance, a rubber ball 25 cm in diameter differs from a cast-iron ball of the same diameter in weight, colour, hardness, etc. Geometry disregards all these qualities of the balls and states that their spatial properties (shape and dimensions) are the same. From the viewpoint of geometry, both objects represent a *sphere* of diameter 25 cm.

An object conceptually stripped of all properties except its spatial qualities is called a *geometric body* (solid). A sphere is a geometric solid.

Extending the process of abstraction, we get the concepts of a geometric *surface*, a geometric *line* and a geometric *point*. We mentally separate the surface of a solid from the body it belongs to and divest it of thickness. We deprive a line of thickness and width. A point is without any dimensions whatever. We conceive of a point as serving as the boundary of a line (or of a part of it), a line, as a boundary of a surface, and a surface, as a boundary of a solid. We visualize the point as moving and, in this motion, as generating a line; a line in motion generates a surface, a moving surface generates a solid.

There are no points in nature devoid of dimensions, but there are objects so small that they can conveniently be taken for geometric points. Neither are there any geometric lines or geometric surfaces in nature, but all the properties of lines and surfaces revealed in geometry find multifarious applications in science and engineering. This is due to the fact that geometric concepts are products of the spatial properties of the real world about us. It is the abstract form of geometric concepts which serves to strip these properties of all inessentials and exhibit them in pure form.

139. Historical Survey of the Development of Geometry

The notions of geometry originated in remote antiquity. They arose out of the necessity to determine the volumes of various objects (vessels, granaries and the like). The most ancient written records which contain rules for determining areas and volumes were compiled in Egypt and Babylonia about 4000 years ago. About 2500 years ago the Greeks took over from the Egyptians and Babylonians their geometric findings. This knowledge was first used mainly in measuring land areas, whence the Greek term "geometry", which means "earth measurement".

The Greek scholars discovered numerous geometric properties and set up a harmonious system of geometric knowledge. For the basic starting principles they took the most elementary geometric properties suggested by experience. The other properties were derived from these by logical reasoning.

About 300 B.C. this system took on a finished form as the "Elements" of Euclid, a work which also includes the essentials of theoretical arithmetic. The geometric sections of the "Elements" hardly differ either in content or rigour from present-day school textbooks of geometry.

But we find nothing there about volume, the surface of a sphere, or the ratio of the circumference of a circle to the diameter (although there is a theorem stating that the areas of circles are in the ratio of the squares of the diameters). The approximate value of this ratio was known from experience long before Euclid's time, but it was only in the middle of the third century B.C. that Archimedes (287—212) gave a rigorous proof that the ratio of the circumference of a circle to the diameter (our number π) lies between $3\frac{1}{7}$ and $3\frac{10}{71}$. Archimedes also proved that the volume of a sphere is less than the volume of a circumscribed cylinder by exactly $1\frac{1}{2}$ times and that the surface of a sphere is $1\frac{1}{2}$ times less than the total surface of a circumscribed cylinder.

The methods used by Archimedes in the solution of the problems mentioned above contain the embryo of the methods of higher mathematics. Archimedes applied these methods to the solution of many difficult problems in geometry and mechanics that were very important to construction work and navigation. For instance, he determined the volumes and the centres of gravity of many solids and studied the question of the equilibrium of floating bodies of diverse shapes.

The Greek geometers investigated the properties of many curves that have theoretical and practical importance. They made a particularly deep study of *conic sections* (see Sec. 167).

In the second century B.C., Apollonius enriched the theory of conic sections with many important discoveries that have remained unsurpassed for eighteen centuries.

In his study of conic sections Apollonius made use of the method of coordinates (see Sec. 211). This method was applied to curves in the plane only in the 1630's by the French mathematicians Fermat (1601—1655) and Descartes (1596—1650). The engineering work of that day did not demand anything beyond plane curves. It was only one hundred years later, when the demands of developing astronomy, geodesy and mechanics became urgent, that the coordinate method was applied to the study of curved surfaces and lines drawn on curved surfaces.

A systematic development of the coordinate method in space was given in 1748 by the great Euler.

For over two thousand years the system of Euclidean geometry was regarded as immutable. But in 1826 the celebrated Russian mathematician N. I. Lobachevsky created a new geometric system. The starting propositions of this system differ from those of Euclid in one point only, yet this difference leads to a multitude of very essential peculiarities. (In Euclidean geometry, through a point *A* there can pass only one straight line in the same plane as a given straight line *BC* without intersecting it. In the geometry of Lobachevsky there are infinitely many such straight lines).

Thus, in Lobachevskian geometry the sum of the angles of a triangle is always less than 180° (in Euclidean geometry it is equal to 180°), and the defect (the amount less than 180°) is the greater, the larger the area of the triangle. It might appear that practical experience refutes this and other conclusions made by Lobachevsky. But this is not so. Measuring the angles of a triangle directly, we find their sum to be approximately 180°. The exact value eludes us because of the imperfections of our measuring instruments. On the other hand, all the triangles that we can measure are too small for us to notice the defect in the sum of the angles via direct measurements.

Subsequent development of Lobachevsky's ideas showed that the Euclidean system is insufficient for a study of many problems of astronomy and physics where we have to do with bodies of immense proportions. However, it suffices fully to handle all practical problems of our mundane life. And since, besides, it has the advantage of simplicity, it will continue to be used in engineering calculations and it will continue to be studied at school.

140. Theorems, Axioms, Definitions

An argument that establishes some property is called a *proof* or demonstration. The property demonstrated is called a *theorem*. In the proof of geometric theorems we proceed from earlier established properties, some of which in turn are theorems; however some are considered in geometry to be basic and are assumed to be true without proof. Such properties, or self-evident truths, are called *axioms*.

Axioms originated from experience, and it is experience again that verifies the truth of axioms as a system. This verification consists in the fact that all the theorems of geometry are in agreement with experiment, which would not be the case if the system of axioms were false.

Not a single geometric property taken separately is an axiom, since it can always be demonstrated on the basis of other properties. For instance, the geometric axiom on the property of parallel lines reads: "only one line can be drawn parallel to a given line through a given point not on this line" (the axiom of parallels). Proof is given, on the basis of this axiom (and a number of others), that the "sum of angles of a triangle is 180°". Yet we could take this latter property for the axiom in place of the axiom of parallels (leaving the other axioms unchanged). Then the property of parallel straight lines may be demonstrated thus becoming a theorem.

Thus, a system of axioms can be chosen in a variety of ways. The sole requirement is that the set of axioms be sufficient to derive all the other geometric properties. In geometry, the attempt is made to reduce the number of axioms as far as possible. This is done in order to elucidate the logical relationships between the separate properties.

Axioms are preferably chosen from among the most elementary geometric properties. True, opinions may differ as to the simplicity of a property.

Some of the concepts of geometry are taken to be initial and their content is extracted from experience alone (such, for instance, is the notion of a point). All other concepts are derived from these initial ones. They are defined and such explanations are called *definitions*. Every geometric definition either stems directly from the initial concepts or is based on earlier defined concepts.

One and the same geometric concept may be defined differently. For example, the diameter of a circle may be defined as a chord that passes through the centre, or as the chord of greatest length. We can take one of these properties as the definition and then prove the other property. Preference is given to the most elementary property; here too, incidentally, it is impossible to ensure complete and general agreement.

141. Straight Line, Ray, Line-Segment

A straight line can be mentally extended in either direction indefinitely. A straight line that is bounded on one side is called a *half-line*, or a *ray*. A straight line bounded on both sides is called a *line-segment*.

142. Angles

An angle (symbol: $\angle$) is a figure (Fig. 59) formed by two rays, OA and OB (the *arms* or *sides* of the angle), emanating from a single point O (*vertex* of the angle).

An angle is measured by the amount of rotation about the vertex O which carries ray OA to OB. Wide use is made of two systems of measuring angles: in terms of *radians* and *degrees*. They differ in the choice of the unit of measurement. Radian measure is discussed in Sec. 180.

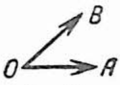

Fig. 59.

*Measurement of angles in degrees.** The unit here is one degree = 1/360 of one complete rotation (symbol: °). Thus, one complete rotation (exemplified by the hand of a clock moving from 0 hrs to 12 hrs) constitutes 360°. A degree consists of 60 minutes (denoted′); a minute consists of 60 seconds (″). For example, 42°33′21″ is read as "42 degrees, 33 minutes, 21 seconds".

An angle of 90° (i. e., one fourth of a complete revolution) is called a right angle (Fig. 60) and is denoted by the letter d.

---

* The degree unit of angular measure goes back to remote antiquity (see Sec. 21, Item 4). During the first French bourgeois revolution of 1793, a centesimal system of angular measurement was introduced in keeping with the decimal (metric) system of measures which was introduced at that time. In that system a right angle was divided into 100 degrees (called grades), one degree into 100 minutes, and one minute into 100 seconds. This system is still used today but not widely, mostly in geodetic measurements.

An angle less than 90° is called *acute* (*AOB* in Fig. 59), an angle greater than 90° is called *obtuse* (Fig. 61). The straight lines that form a right angle are called *perpendicular lines*.

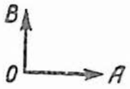

Fig. 60.

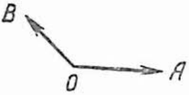

Fig. 61.

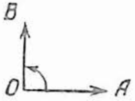

Fig. 62.

Signs of angles. It is often important to know in what direction a ray is being rotated. In angular measurement, it is ordinarily taken that counterclockwise rotation of a ray corresponds to *positive* values, while clockwise rotation indicates *negative* values. For example, if the ray *OA* moves to coincidence with *OB*, as indicated in Fig. 62, then $\angle AOB = +90°$. In Fig. 63, $\angle AOB = -90°$. In Fig. 64, $\angle AOB = -270°$. One and the same arrangement of rays may correspond to different angular measures depending on the type of rotation. Thus, $\angle AOB$ in Fig. 65 may be taken equal to $+450°$. In elementary geometry, angular measurements are always taken to be positive and are considered to measure the smallest rotation so that one does not measure angles greater than 180°.

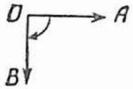

Fig. 63.

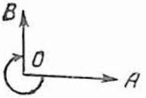

Fig. 64.

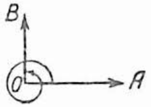

Fig. 65.

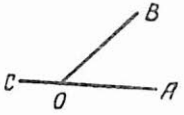

Fig. 66.

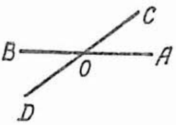

Fig. 67.

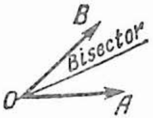

Fig. 68.

Adjacent angles. In Fig. 66 the pair of angles *AOB* and *COB* with common vertex *O* and common side *OB* are adjacent angles. The two other sides, *OA* and *OC* are extensions of one another. The sum of these two adjacent angles is 180° ($2d$). In Fig. 70 we have adjacent angles AON and NOB which are less than 180°. Angles which have a common vertex, one common arm and are on opposite sides of the common arm, are called adjacent angles.

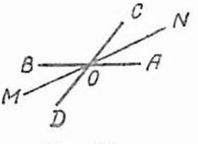

Fig. 69.

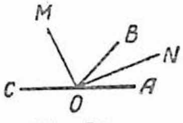

Fig. 70.

Vertically opposite angles (or simply, vertical angles) are those which have a common vertex and the sides of one are the extended sides of the other. In Fig. 67, $\angle AOC$ and $\angle DOB$ (also $\angle COB$ and $\angle AOD$) are vertical angles. Vertical angles are equal ($\angle AOC = \angle BOD$).

When speaking of "*an angle between two straight lines*", we mean any one of the four angles produced (usually the acute angle).

The *bisector* of an angle is the line that divides the angle into two equal parts (Fig. 68). The bisectors of vertical angles (OM and ON in Fig. 69) form a single straight line. The bisectors of adjacent supplementary angles are perpendicular to each other (Fig. 70).

143. Polygons

A plane figure formed by a closed series of rectilinear segments is called a *polygon*. Figure 71 depicts a hexagon $ABCDEF$. The points A, B, C, D, E, F, are the *vertices of the polygon*; the angles at these points (the angles of the polygon) are denoted by $\angle A$, $\angle B$, $\angle C, \ldots, \angle F$. The lines AC, AD, BE, etc. are *diagonals*; AB, BC, CD, etc. are the *sides of the polygon*. The sum of the lengths of the sides, $AB + BC + CE + \ldots + FA$, is termed the *perimeter* and is denoted by p, sometimes $2p$ (then p is the semiperimeter).

Only *simple* polygons (that is, such that the contour has no self-intersections) are studied in elementary geometry. Polygons whose contours have self-intersections are called *star polygons*. A star polygon $ABCDE$ is shown in Fig. 72.

If all the diagonals of a polygon lie inside the figure, the polygon is termed *convex*. The hexagon shown in Fig. 71 is convex; the pentagon in Fig. 73 is nonconvex, or concave (the diagonal EC lies without the polygon).

The sum of the interior angles in any convex polygon is equal to $180° (n-2)$, where n is the number of sides of the polygon. *

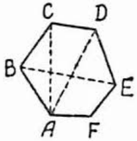

Fig. 71.

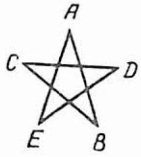

Fig. 72.

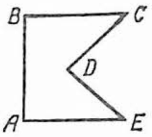

Fig. 73.

144. Triangles

A *triangle* (symbols: $\triangle$, plural $\triangle$s) is a polygon having three sides. The sides of a triangle are frequently denoted by lower-case letters corresponding to the labels of the opposite vertices. If all three angles of a triangle are acute, the figure is called an *acute-angle triangle* (Fig. 74); if one of the angles is a right angle, it is called a *right triangle* (Fig. 75); the sides forming the right angle are called *legs* (a, b); the side opposite the right angle is called the *hypotenuse* (c). If one of the angles is obtuse (say, $\angle B$ in Fig. 76), then we have an *obtuse-angle triangle*.

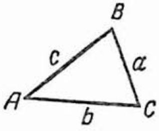

Fig. 74.

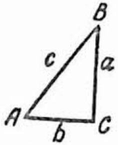

Fig. 75.

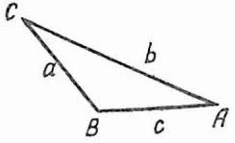

Fig. 76.

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* In geometry textbooks this property is ordinarily stated only for convex polygons, yet it is valid for all "simple" polygons. Note that in a concave polygon, one or several interior angles exceed 180°. Thus, in the concave pentagon shown in Fig. 73, two angles are right angles, two are 45° angles, and one contains 270°. The sum of the angles is $180° (5-2)=540°$.

Triangle ABC (Fig. 77) is an *isosceles* triangle: two of its sides are equal ($b=c$). When three sides are equal ($a=b=c$), as in Fig. 78, we have an *equilateral triangle*.

In any triangle, the greater angle is opposite the greater side; if the sides are equal, the opposite angles are equal; and conversely, if the angles are equal, the opposite sides are equal. In particular, equilateral triangles are also equiangular, and conversely.

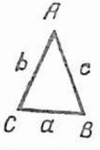

Fig. 77.

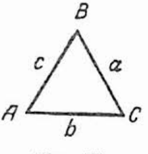

Fig. 78.

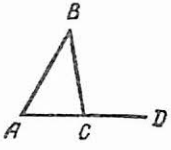

Fig. 79.

The sum of the angles of any triangle is 180°. Each angle in an equilateral triangle is equal to 60°.

Producing one of the sides of a triangle (AC in Fig. 79), we get an *exterior angle*, $\angle BCD$. *An exterior angle is equal to the sum of the nonadjacent interior angles:* $\angle BCD = \angle A + \angle B$.

Any side of a triangle is less than the sum and more than the difference of the other two sides ($a < b+c$; $a > b-c$).

The area of a triangle is equal to one-half the product of the base and the altitude (see Sec. 146 on the altitude of a triangle): $S = 1/2\, ah_a$.

145. Congruence of Triangles

Two triangles are congruent if the following elements are respectively equal:

(1) two sides and the included angle; for example, $AB=A'B'$; $AC=A'C'$; $\angle A = \angle A'$ (Fig. 80);

(2) two angles and the adjacent side; for example, $\angle A = \angle A'$; $\angle C = \angle C'$; $AC = A'C'$;

(2a) two angles and the side opposite one of them, for example, $\angle A = \angle A'$; $\angle B = \angle B'$; $AC = A'C'$;

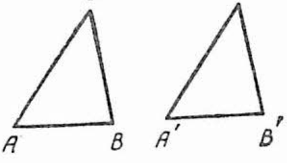

Fig. 80.

(3) three sides: $AB = A'B'$; $BC = B'C'$; $AC = A'C'$;

(4) two sides and the angle opposite the greater side; for example, $AB = A'B'$, $BC = B'C'$, $\angle A = \angle A'$ in Fig. 80, where BC is greater than AB. If equal angles lie opposite smaller sides, then the triangles may not be congruent. For example, the triangles LMN and $L'M'N'$ in Fig. 81 are not congruent although $LM = L'M$ and $LN = L'N'$ and $\angle M$ $\angle M'$. Here, the angles M, M' lie opposite the smaller sides LN, $L'N'$.

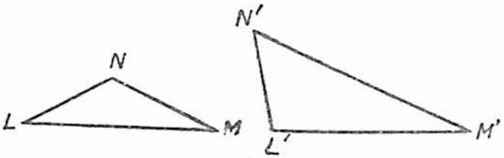

Fig. 81.

146. Remarkable Lines and Points of the Triangle

The *altitude* (or height) of a triangle is the perpendicular drawn from any vertex of the triangle to the opposite side or its extension (the side to which the perpendicular is drawn is then called the *base* of the triangle). In an obtuse-angle triangle (ABC, in Fig. 82), two altitudes (AD, BE) fall on the extensions of the sides, outside the triangle, and the third altitude (CF) falls inside the triangle. In an acute-angle triangle (Fig. 83), all three altitudes lie within the triangle. In a right triangle, the legs serve as altitudes. The three altitudes of a triangle always meet in one point called the *orthocentre*. In an obtuse-angle triangle the orthocentre lies

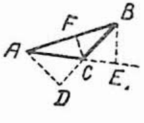

Fig. 82.

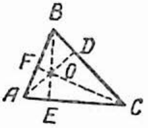

Fig. 83.

outside the triangle, in a right triangle it coincides with the vertex of the right angle.

The altitude of a triangle dropped onto side a is denoted by h_a. It is expressed in terms of the three sides by the following formula:

$$h_a = \frac{2\sqrt{p(p-a)(p-b)(p-c)}}{a}$$

where

$$p = \frac{a+b+c}{2}$$

Median. A straight line which joins a vertex of a triangle to the midpoint of the opposite side is called a median. The three medians of a triangle (AD, BE, CF in Fig. 84) are

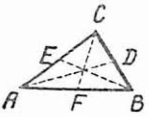

Fig. 84.

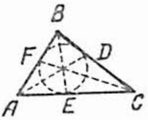

Fig. 85.

concurrent (that is, intersect in one point: the point is always inside the triangle). This point is the centroid (the centre of gravity of the triangle). It divides each median in the ratio 2:1 (reckoning from the vertex). The median which joins the vertex A of a triangle to the midpoint of side a is denoted by m_a. It is given, in terms of the sides of the triangle, by the formula

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$

An *angle bisector* of a triangle (see Sec. 142) is the line segment from the given vertex to its intersection with the opposite side. The three angle bisectors of a triangle (AD, BE, CF, in Fig. 85) intersect in one point (always inside the triangle) which is the centre of the inscribed circle (see Sec. 156). The bisector of angle A is denoted by β_a. Its length is given, in terms of the sides of the triangle, by the following formula:

$$\beta_a = \frac{2}{b+c}\sqrt{bcp(p-a)}$$

where p is the semiperimeter. A bisector divides the opposite side into parts proportional to the sides adjacent to it. In Fig. 85, $AE:EC = AB:BC$.

Example. $AB = 30$ cm, $BC = 40$ cm, $AC = 49$ cm. Find AE and EC. The two parts (AE and EC) into which $AC = 49$ cm is to be divided are in the ratio 30:40 or 3:4. Taking x as the scale unit ($x = {}^1/_3\,AE$, $x = {}^1/_4\,EC$), we have $AC = 3x + 4x = 7x$, $x = AC:7 = 49:7 = 7$, whence $AE = 3x = 21$; $EC = 4x = 28$.

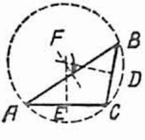

Fig. 86.

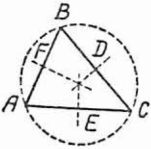

Fig. 87

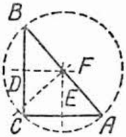

Fig. 88.

The three perpendicular lines drawn to the sides of a triangle through their midpoints (D, E, F, in Figs. 86, 87, 88) meet at one point which is the circumcentre (see Sec. 156). In an obtuse-angle triangle (Fig. 86) this point lies outside the triangle, in an acute-angle triangle (Fig. 87) it lies inside the triangle, in a right triangle it lies at the midpoint of the hypotenuse (Fig. 88).

In an isosceles triangle, the altitude, the median, the angle bisector and the perpendicular drawn to the midpoint of the base (which in an isosceles triangle is always the side different from the other two) all coincide. The same situation holds true for all three sides of an equilateral triangle. In all other cases none of these lines coincides with any other one. The orthocentre, the centroid, the centre of the inscribed circle and the circumcentre coincide only in an equilateral triangle.

147. Orthogonal Projections. Relationships Between the Sides of a Triangle

The *orthogonal projection* (or, briefly, *projection*) of a point on a straight line is the foot of the perpendicular from the point to the line. In Fig. 89, the points a, b, c, d are projections of the points A, B, C, D on the straight line MN. The projection of AB on MN is the line segment ab of the straight line MN bounded by the projections a and b of the endpoints of AB. The line segment bc is the projection of BC, etc. This is denoted as $ab = \text{pr}_{MN}AB$ or, briefly, $ab = \text{pr}\, AB$.

The sum of the projections of the segments of a polygonal line is equal to the projection of the closing segment. In Fig. 89, $\text{pr}\, AD = \text{pr}\, AB + \text{pr}\, BC + \text{pr}\, CD$. For complete generality, we must regard the projection of a line as an algebraic quantity; the projection ab of segment AB is taken to be *positive* if b is to the right of a, and *negative* if b is to the left of a. Thus, in Fig. 90, $\text{pr}\, AB = ab$ is negative; $\text{pr}\, BC = bc$, $\text{pr}\, CD = cd$, $\text{pr}\, DE = de$ are positive; $\text{pr}\, EF = ef$ is negative. Therefore, the (algebraic) sum of the projections of the segments of the polygonal line $ABCDEF$ is obtained by adding the lengths of the segments bc, cd, de and subtracting the sum of the lengths of the segments ab and ef. The result is equal to af, which is the projection of the closing segment AF.

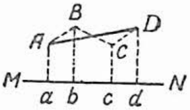

Fig. 89.

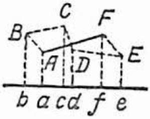

Fig. 90.

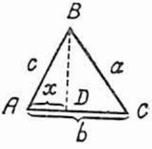

Fig. 91.

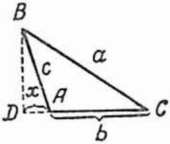

Fig. 92.

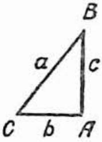

Fig. 93.

The square of a side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of one of the two sides by the projection, on it, of the other. Using the notations of Figs. 91 and 92, we have

$$a^2 = b^2 + c^2 - 2b \cdot \text{pr}_{AC}AB \tag{1}$$

If x denotes the length of a projection (a positive number), then, when angle A is acute ($\text{pr}_{AC}AB = x$ in Fig. 91),

$$a^2 = b^2 + c^2 - 2bx \tag{2}$$

and when angle A is obtuse ($\mathrm{pr}_{AC}AB = -x$ in Fig. 92),

$$a^2 = b^2 + c^2 + 2bx \tag{3}$$

If angle A is a right angle (Fig. 93), then $\mathrm{pr}_{AC}AB = 0$ and we have

$$a^2 = b^2 + c^2 \tag{4}$$

the square of the hypotenuse is equal to the sum of the squares of the legs (this is known as the Pythagorean theorem *). The theorem of Pythagoras finds extensive applications in both practical and theoretical situations.

Formula (1) can also be written as

$$a^2 = b^2 + c^2 - 2bc\cos A$$

(see Sec. 199).

148. Parallel Straight Lines

Two straight lines AB and CD (Fig. 94) are called *parallel lines* if they lie in one plane and do not meet however far produced. Symbol: $AB \parallel CD$. The distance between the lines is everywhere the same.

Fig. 94.

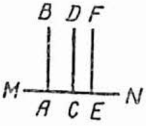

Fig. 95.

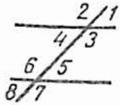

Fig. 96.

All straight lines parallel to the straight line AB are parallel among themselves.

It is considered that two parallel straight lines form an angle equal to zero (there is no angle at all in the direct sense of the word).

If two rays belong to parallel straight lines, then the angle between the rays is taken to be zero when the direction of the rays is the same, and 180° when the directions of the rays are opposite.

Straight lines (AB, CD, EF, Fig. 95) perpendicular to a single straight line MN are themselves parallel. Conversely, the straight line MN perpendicular to one of the parallel lines is perpendicular to all others. All lines perpendicular to one of two parallel straight lines are perpendicular to the other. The lengths of these perpendiculars between the two parallel lines are equal. Their common length is the *distance* between the parallel straight lines.

---

* The theorem is credited to Pythagoras, a Greek philosopher who lived in 6th and 5th centuries B. C. Actually this theorem was known in the Ancient East 20 centuries before the Christian Era.

A straight line (called a transversal) that cuts two parallel straight lines forms eight angles (Fig. 96), pairs of which have the following names: (1) corresponding angles (1 and 5; 2 and 6; 3 and 7; 4 and 8); these angles are pairwise equal: $\angle 1 = \angle 5$; $\angle 2 = \angle 6$; $\angle 3 = \angle 7$; $\angle 4 = \angle 8$;

(2) alternate interior angles (4 and 5; 3 and 6); they are pairwise equal;

(3) alternate exterior angles (1 and 8; 2 and 7), they are also pairwise equal;

(4) interior angles on the same side of the transversal (3 and 5; 4 and 6); the sum of these angles is equal to 180° ($\angle 3 + \angle 5 = 180°$; $\angle 4 + \angle 6 = 180°$);

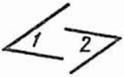

Fig. 97.

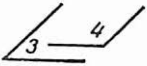

Fig. 98.

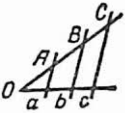

Fig. 99.

(5) exterior angles on the same side of the transversal (1 and 7; 2 and 8); the sum of these angles is equal to 180° ($\angle 1 + \angle 7 = 180°$; $\angle 2 + \angle 8 = 180°$).*

Angles *with corresponding sides parallel* are either equal (if both are acute or both are obtuse) or their sum is equal to 180°; in Fig. 97 $\angle 1 = \angle 2$; in Fig. 98, $\angle 3 + \angle 4 = 180°$. Angles with corresponding sides perpendicular are likewise either equal or constitute a total of 180°.

Parallel straight lines that intersect the sides of an angle, as shown in Fig. 99, intercept proportional lengths on the sides of the angle:

$$\frac{OA}{Oa} = \frac{OB}{Ob} = \frac{OC}{Oc} = \frac{AB}{ab} = \frac{BC}{bc} = \frac{AC}{ac}, \text{ etc.}$$

---

* When two nonparallel straight lines are cut by a transversal the angles formed bear the same names as those given above, but the relationships between the angles no longer hold true.

149. The Parallelogram and the Trapezoid

A *parallelogram* ($ABCD$ in Fig. 100) is a quadrilateral in which both pairs of opposite sides are parallel. The opposite sides of a parallelogram are equal: $AB=CD$, $AD=BC$. Any two opposite sides may be taken as *bases*. The perpendicular distance between them is called the *altitude* (BF). The diagonals of a parallelogram bisect each other ($AO=OC$, $BO=OD$). The opposite angles of a parallelogram are equal ($\angle A=\angle C$, $\angle B=\angle D$). *The sum of the squares of the diagonals is equal to the sum of the squares of the four sides:* $AC^2+BD^2=AB^2+BC^2+CD^2+AD^2=2(AB^2+BC^2)$. The area S of a parallelogram is equal to the product of the base (a) by the altitude (h_a):

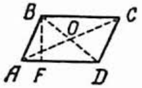

Fig. 100.

$$S=ah_a$$

Distinguishing features of parallelograms. A quadrilateral $ABCD$ is a parallelogram provided that:

(1) the opposite sides are equal ($AB=CD$, $BC=DA$);

(2) two opposite sides are equal and parallel

($AB=CD$, $AB \parallel CD$);

(3) the diagonals bisect each other;

(4) opposite angles are equal ($\angle A=\angle C$, $\angle B=\angle D$).

If one of the angles of a parallelogram is a right angle, then all angles are right angles. Such a parallelogram is called a *rectangle* (Fig. 101). The sides of a rectangle (a, b) serve as its altitudes. *The area of a rectangle is equal to the product of its sides:* $S=ab$.

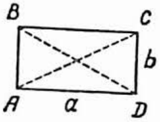

Fig. 101.

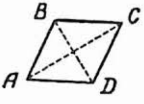

Fig. 102.

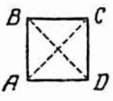

Fig. 103.

The diagonals of a rectangle are equal: $AC=BD$.

In a rectangle, the square of a diagonal is equal to the sum of the squares of the sides: $AC^2=AD^2+DC^2$.

If a parallelogram has all sides equal, it is called a *rhombus* (Fig. 102).

In a rhombus, the diagonals are mutually perpendicular ($AC \perp BD$) and bisect the angles ($\angle DCA = \angle BCA$, etc.).

The area of a rhombus is equal to half the product of the diagonals:

$$S = \frac{1}{2} d_1 \cdot d_2$$

where $d_1 = AC$, $d_2 = BD$.

A *square* is a parallelogram with right angles and equal sides (Fig. 103). A square is a particular type of rectangle and also a particular type of rhombus. It therefore has all their properties, as given above.

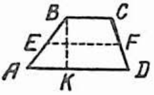

Fig. 104.

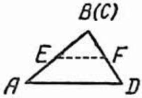

Fig. 105.

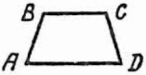

Fig. 106.

A *trapezoid* is a quadrilateral having one pair of opposite sides parallel ($BC \parallel AD$, Fig. 104). A parallelogram may be considered as a particular type of trapezoid.

The parallel sides are called the *bases* of the trapezoid, the other two sides (AB, CD) are called its *nonparallel sides.* The perpendicular distance (BK) between the bases is called the *altitude* of the trapezoid. The line EF joining the midpoints of the nonparallel sides is called the *median* (or *midline*) of the trapezoid.

The median of a trapezoid is equal to one-half the sum of the bases: $EF = \frac{1}{2}(AD + BC)$, *and is parallel to them:* $EF \parallel AD$.

The area of a trapezoid is equal to the product of the median by the altitude:

$$S = \frac{1}{2}(a + b)h$$

where $a = AD$, $b = BC$, $h = BK$.

A triangle is the limiting (degenerate) case of a trapezoid when one of the bases shrinks to a point (Fig. 105). A degenerate trapezoid preserves its properties; for example, the line joining the midpoints E and F of the sides of triangle ABD

(the median of the triangle) is parallel to side AD and is equal to one-half of it.

A trapezoid with equal nonparallel sides is called an *isosceles trapezoid* ($AB = CD$ in Fig. 106). The base angles of an isosceles trapezoid are equal ($\angle A = \angle D$, $\angle B = \angle C$).

150. Similarity of Plane Figures. Similar Triangles

If all the dimensions of a plane figure are changed (increased or decreased) in the same ratio (the *ratio of similitude*), then the old and new figures are called *similar figures*. For instance, a picture and a photograph of the picture.

In two similar figures, any corresponding angles are equal; that is, if points A, B, C, D in one figure correspond to points a, b, c, d of another, then $\angle ABC = \angle abc$, $\angle BCD = \angle bcd$, etc.

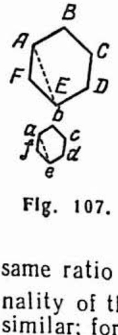

Fig. 107.

Two polygons ($ABCDEF$ and $abcdef$ in Fig. 107) are similar if they have equal angles ($\angle A = \angle a$, $\angle B = \angle b$, ..., $\angle F = \angle f$) and their corresponding sides are proportional $\left(\frac{AB}{ab} = \frac{BC}{bc} = \frac{CD}{cd} = \ldots = \frac{FA}{fa}\right)$. This ensures the proportionality of all other corresponding elements of the polygons; for example, the diagonals AE and ae have the same ratio as the sides $\left(\frac{AE}{ae} = \frac{AB}{ab}\right)$. However, the proportionality of the sides of polygons does not suffice to make them similar; for example, in Fig. 108 the sides of the quadrilateral $ABCD$ (square) are proportional to the sides of the quadrilateral $abcd$ (rhombus); each side of the square is twice that of the rhombus. But the diagonals of the square did not diminish in the same proportion (one diminished more than twice, the other, less than twice) because the angles of the rhombus $abcd$ are not equal to the angles of the square $ABCD$.

In the similarity of triangles, on the other hand, the proportionality of the sides is sufficient: *two triangles are similar if their sides are proportional*. Thus, if the sides of a triangle ABC (Fig. 109) are twice the length of the sides of triangle abc, then the angle bisector BD is twice the bisector bd, the altitude BE is twice the altitude be, etc., and the corresponding angles are equal ($\angle A = \angle a$, $\angle B = \angle b$, $\angle C = \angle c$).

If the corresponding angles of two triangles are equal, the triangles are similar (it is sufficient to detect the equality of two pairs of angles because the sum of the angles of a triangle is always 180°). This criterion is not sufficient for arbitrary polygons. For example, the square $ABCD$ and the rectangle $abcd$ (Fig. 110) have equal corresponding angles, but the figures are not similar.

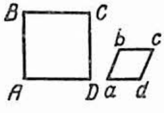

Fig. 108.

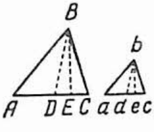

Fig. 109.

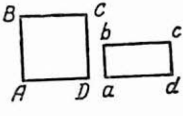

Fig. 110.

Triangles are also similar when two sides of one are proportional to two sides of the other and the included angles are equal (that is, if $\frac{AB}{ab}=\frac{BC}{bc}$ and $\angle B=\angle b$).

Right triangles are similar if the hypotenuse and a leg of one are proportional to the hypotenuse and a leg of the other.

Any two circles are similar, one being an increased or decreased version of the other.

The areas of similar figures (say, polygons) *are proportional* to the squares of their corresponding lines (sides, for example). In particular, the areas of circles stand in the same proportion to one another as the squares of the radii or the diameters. It would therefore be a serious mistake to consider that the ratio of the areas of two circles is equal to the ratio of their diameters. This mistake is often made, however.

Example 1. A circular metal disk of diameter 20 cm weighs 2.4 kg. How much does a disk (of the same material and thickness) 10 cm in diameter weigh?

To reason that the diameter of the small disk is one-half that of the large disk and on that basis to say that the small disk weighs one-half the large disk, or 1.2 kg, would be a grave mistake.

Here is the correct solution. Since the material and thickness of the disk remain the same, the weights are proportional to the areas, and the ratio of the area of the small disk to the area of the large disk is $\left(\frac{10}{20}\right)^2=\frac{1}{4}$. Hence, the weight of the small disk is $2.4\cdot\frac{1}{4}=0.6$ kg.

Example 2. The population of a country, call it A, is given as 8.2 million, that of some country B, 4.1 million. If the population of B is shown diagrammatically as a square with side 10 cm, what will the side of the square depicting the population of A be?

Denoting the required side by a, we have

$$\left(\frac{a}{10}\right)^2 = \frac{8.2}{4.1} = 2; \quad \frac{a}{10} = \sqrt{2} \approx 1.4; \quad a \approx 14 \text{ cm}$$

151. Loci and Circles

The *locus* (plural, loci) of points having a given property is the totality of the points satisfying the given conditions.

The circle is a locus of points in the plane that are equidistant from one point, the centre.

Equal line-segments joining the centre to points of the circumference are called *radii*, (singular, *radius*, denoted by r or R). Part of the circumference (say AmD in Fig. 111) is called an *arc*. The straight line MN passing through two points of the circumference is called a *secant*; the portion of it, KL, lying inside the circle is a *chord*. A chord increases as it approaches the centre of the circle. The chord BD passing through the centre (O) is called the *diameter*, denoted by d or D. The diameter is equal to two radii ($d = 2r$).

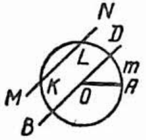

Fig. 111.

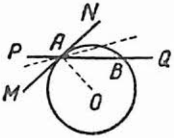

Fig. 112.

A *circle* is a plane curve consisting of all points at a given distance (called the radius) from a fixed point in the plane, called the centre. A circle also denotes the region of a plane all points of whose boundary are at a given distance (called the radius) from a fixed point in the plane (called the centre).

Tangent line. Let the secant PQ (Fig. 112) pass through the points A and B of the circle. Let the point B move along the circumference approaching A. The secant line PQ will change its position as it rotates about A. As B approaches A, the secant PQ will tend to a certain limiting position MN. The straight line MN is called the *tangent* to the circle at the point A. The tangent and the circle have only one point in common.* We may consider a tangent to be a degenerate secant.

A tangent to a circle is perpendicular to the radius OA drawn to the point A of contact.

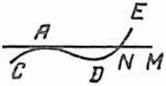

Fig. 113.

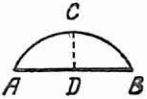

Fig. 114,

From a point without a circle we can draw two tangents to the circle; the tangents will be of equal length (see Fig. 120).

A *segment* of a circle is the area between a chord and the arc subtended by the chord (in Fig. 114 the arc ACB subtended by the chord AB).

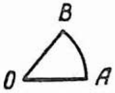

Fig. 115.

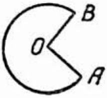

Fig. 116.

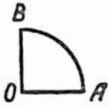

Fig. 117.

The perpendicular drawn from the midpoint of the chord AB to intersection with the arc AB is called the *sagitta* of the arc AB. The length of the sagitta DC (Fig. 114) is the *altitude* of the segment.

A *sector* of a circle is that portion of the circle bounded by two radii of the circle and one of the arcs which they intercept (Figs. 115 and 116). A sector with radii that form an angle of 90° is called a *quadrant* (Fig. 117).

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[*] This property is ordinarily taken as a definition of a tangent to a circle. For other lines, however, this definition may prove invalid. For example, MN in Fig. 113 is tangent to the line $CADE$ at point A, but MN has, besides A, a point N that belongs to $CADE$ as well. The definition we give of a tangent as the limiting position of a secant is applicable to all lines.

152. Angles in a Circle. The Length of the Circumference and of an Arc

A *central angle* is formed by two radii (∠ AOB in Fig. 118).

An *inscribed angle* is formed by two chords (CA and CB in Fig. 119) issuing from a common point of the circle (∠ ACB in Fig. 119).

A *circumscribed angle* is one formed by two tangents issuing from one point (CA and CB which form ∠ ACB in Fig. 120).

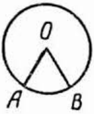


Fig. 118.

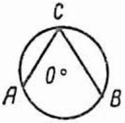


Fig. 119.

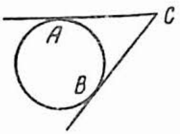


Fig. 120.

The *length of an arc* described by the endpoint of a radius is proportional to the magnitude of the corresponding central angle; for this reason, the arcs of one and the same circle may be measured like angles, by degrees (see Sec. 142). Namely, 1° of arc is taken to be $\frac{1}{360}$ of the circle (that is, an arc whose central angle is equal to 1°). The entire circle contains 360°, one half is 180°.

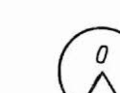


Fig 121.

To avoid errors that occur frequently, note that the value of the central angle is quite independent of the length of the radius whereas the magnitude of the corresponding arc is proportional to the radius. Thus, in Fig. 121, the central angle preserves the same value irrespective of whether we form it with the radii OC and OD or OA and OB, which are half that length. Now the arcs AB and CD are of unequal length although they have the same number of degrees: arc AB is shorter than arc CD.

Generally, the length of an arc is proportional to (1) the radius and (2) the magnitude of the corresponding central angle.

The length of the circumference p constitutes about $3\frac{1}{7}$ the length of the diameter: $p \approx 3\frac{1}{7}d$. In other words, the ratio of the lengths of circumference and diameter is approximately $3\frac{1}{7}$:

$$\frac{p}{d} \approx 3\frac{1}{7}$$

The exact ratio $\frac{p}{d}$ is denoted by the Greek letter pi (π):

$$\frac{p}{d}=\pi \tag{1}$$

$3\frac{1}{7}$ is an approximate (too high) value of the number π. The number π is irrational (see Sec. 91), which means it cannot be written exactly as a fraction. To five decimal places, pi is given as 3.14159. For practical purposes it suffices to take the value $\pi \approx 3.14$, which is slightly less (the difference is inessential) than $\pi \approx 3\frac{1}{7}$.

Formula (1) yields

$$p=\pi d \tag{2}$$

or

$$p=2\pi r \tag{3}$$

The length of 1° of arc is

$$p_{1^\circ}=\frac{2\pi r}{360}=\frac{\pi r}{180} \tag{4}$$

An arc length of n° is

$$p_{n^\circ}=\frac{\pi r n}{180} \tag{5}$$

Formulas (2) to (5) — all of them are readily derived from (1) — are of great theoretical and practical value.

Example 1. A 2.4-metre strip of steel is used to make a hoop; 0.2 metre is taken up at the ends for riveting. What is the radius of the hoop?

The length of the circumference $p=2.4-0.2=2.2$ metres. By formula (3)

$$r=\frac{p}{2\pi} \approx \frac{2.2}{6.3} \approx 0.35 \text{ m}$$

Example 2. The diameter of the driving wheel of a locomotive is 1.5 metres. How many rotations per minute does the wheel make when the train's speed is 30 km/hr?

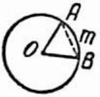
The wheel covers per minute a distance of $30:60=\frac{1}{2}$ (km), or 500 metres. In one rotation, it covers a distance equal to the length of the circumference p; $p=\pi d \approx 3.14\cdot 1.5 \approx 4.71$ metres. The desired number of rotations is $500:4.71 \approx 106$.

Example 3. A railway line has a track curvature of 800 metres radius. The track length on this curvature is 60 metres. How many degrees are there in the arc of the curvature?

By formula (5),

$$n=\frac{180p}{\pi r} \approx \frac{180\cdot 60}{3.14\cdot 800} \approx 4°20' \text{ (rounded)}$$

The *area of a circle* is equal to the product of half the circumference by the radius:



$$S=\frac{1}{2}pr \quad \text{or} \quad S=\pi r^2$$

Fig. 122.

The *area of a sector* of a circle (S_{sect}) is equal to the product of half the arc length (p_{sect}) by the radius (r):

$$S_{sect}=\frac{1}{2}p_{sect}r$$

The area of a sector with an arc of $n°$ is

$$S_{n°}=\frac{\pi r^2 n}{360}$$

The area of a segment of a circle is found as the difference between the area of the sector $AOBm$ (Fig. 122) and the triangle AOB.

152a. Huygens' Formula for Arc Length

In practical situations, one often has to find the length of an arc given in a drawing (or full-size) when it is not known what part of the circle the arc constitutes and what the radius of the circle is. In such cases, use can be made of the following device.

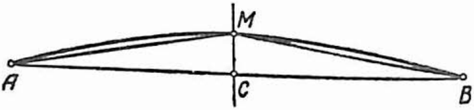


Fig. 122 a.

Mark the midpoint M (Fig. 122a) of a given arc $\overset{\frown}{AB}$ (this point lies on the perpendicular CM drawn to the midpoint C

of chord AB). Then measure the chord AB and the chord AM which intercepts half the arc. The length p of arc $\overset{\smile}{AB}$ is given (approximately) by the following formula of Huygens: *

$$p \approx 2l + \frac{1}{3}(2l - L)$$

where $l = AM$ and $L = AB$.

This formula admits of a relative error of about 0.5%, when $\overset{\smile}{AB}$ subtends 60°. The percent of error falls off sharply as the angular measure of the arc decreases. Thus, for an arc of 45° the relative error constitutes approximately 0.02%.

Example. Figure 122*a* depicts an arc $\overset{\smile}{AB}$ for which

$$l = AM = 34.0\text{ mm}, \quad L = AB = 67.1\text{ mm}$$

Huygens' formula yields

$$p = 2 \cdot 34.0 + \frac{1}{3}(2 \cdot 34.0 - 67.1) \approx 68.3\text{ (mm)}$$

Here, all figures are correct since the arc $\overset{\smile}{AB}$ subtends about 45° (this can be gauged by eye) and, hence, the error of the formula comes to roughly 0.02%, which is less than 0.05 mm.

153. Measuring Angles in a Circle

An *inscribed angle* constitutes one-half the central angle of the intercepted arc. In Fig. 123, $\angle ACB = \frac{1}{2} \angle AOB$.

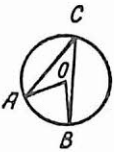


Fig. 123.

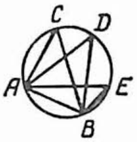


Fig. 124.

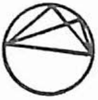


Fig. 125.

Therefore all inscribed angles intercepting the same arc are equal. In Fig. 124, $\angle ACB = \angle ADB = \angle AEB$. In other words, the chord AB is seen from the same angle from all points of the intercepted arc. We say that arc $ACDEB$ subtends an angle of a definite magnitude. For example, a semicircle subtends an angle of 90° (Fig. 125).

---

* Christian Huygens (1629–1695), Dutch scientist noted for works in the fields of optics and mechanics.

Since the central angle contains as many angle degrees as arc degrees, it follows that an *inscribed angle* ($\angle ACB$ in Fig. 123) is measured by one half the intercepted arc AB.

An angle formed by two chords (say, $\angle AOB$ in Fig. 126) is measured by half the sum of the arcs, $\frac{1}{2}(\overset{\smile}{CD}+\overset{\smile}{AB})$, contained between its two sides (produced in both directions). An inscribed angle is a special case of the one under consideration (one of the arcs is equal to zero).

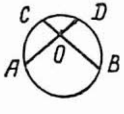


Fig. 126

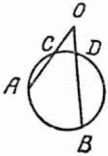


Fig. 127.

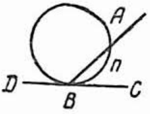


Fig. 128.

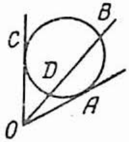


Fig. 129.

An *angle formed by two secant lines* ($\angle AOB$, Fig. 127) is measured by one-half the difference of the arcs lying between its two sides: $\frac{1}{2}(\overset{\smile}{AB}-\overset{\smile}{CD})$. An inscribed angle is a particular case of the angle between two secant lines ($\overset{\smile}{CD}=0$).

Regarding a tangent line as a degenerate secant line (Sec. 151), we find that the *angle formed by a tangent line and a chord* (say, ABC in Fig. 128), is measured by one-half the intercepted arc $\left(\frac{1}{2}\overset{\smile}{AnB}\right)$; an *angle formed by a tangent line and a secant line* (say, $\angle BOA$ in Fig. 129) is measured by one-half the difference of the arcs between its sides: $\frac{1}{2}(\overset{\smile}{BA}-\overset{\smile}{DA})$; a *circumscribed angle* ($\angle COA$ in Fig. 129) is measured by one-half the difference of the arcs between its sides: $\frac{1}{2}(\overset{\smile}{CBA}-\overset{\smile}{CDA})$.

154. Power of a Point

The *power of a point O* with reference to a given circle of radius r is the quantity d^2-r^2, where d is the distance OC from the point to the centre of the circle. The power of a point external to the circle is positive, that of a point internal to the circle, negative. For points on the circumference, the power is zero.

The absolute value of the power of a point, $|d^2-r^2|$, is denoted by p^2 so that for an exterior point, $p^2=d^2-r^2$, and for an interior point, $p^2=r^2-d^2$. The quantities p^2 and p (the latter is assumed to be positive) play an important role.

Let there be drawn through O (Figs. 130 and 131) all kinds of secant lines (AB, DE, FG, etc.). The product of the length of a secant from O to its point of intersection with the circle ($OA \cdot OB$ or $OD \cdot OE$ or $OF \cdot OG$, etc.) is a constant equal to p^2. Of particular importance is the case when the secant line passes through the centre C (see examples below).

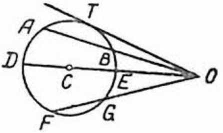


Fig. 130.

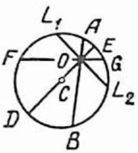


Fig 131.

If point O is external (Fig. 130), then, regarding the tangent line as a degenerate secant line, we have $OT^2=p^2$, or the *absolute value of the power of a point is the square of the length of the tangent line*. Thus, the quantity p is equal to the length of the tangent line OT.

If point O is internal (Fig. 131), then, drawing through O the chord L_1L_2 perpendicular to the diameter DE, we have $OL_1=OL_2$ so that $OL_1^2=p^2$, or the power of a point is equal to the square of the least semichord passing through the point. Thus, the quantity p is equal to the length of the semichord OL_1.

Example 1. What is the range of sight from an aircraft flying over the sea at an altitude of 2 km? Take the diameter of the earth at 12,700 km.

Figure 132 gives a schematic vertical cross section of the earth with O as the location of the aircraft, $OE=2$ km, $ED \approx 12{,}700$ km. The point farthest away on the earth's surface as seen from the airplane is T; OT is the tangent line to the circle ETD; $OT=p$. On the other hand, $p^2=OE \cdot OD \approx 2 \cdot 12{,}700$ (we take $OD \approx 12{,}700$ km, dropping 2 km as a quantity definitely less than the limiting error of the approximate value of 12,700 km). We thus get

$$p=\sqrt{25{,}400} \approx 160 \text{ (km)}$$

Example 2. The span of a masonry vault is 6 metres; it has an altitude of 0.4 metre. Determine the radius of the arc of the vault.

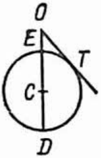


Fig. 132.

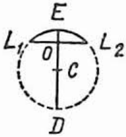


Fig. 133.

See Fig. 133 (schematic): $L_1L_2=6$ m, $EO=0.4$ m. The power of the point O is $p^2=OL_1^2=\left(\frac{L_1L_2}{2}\right)^2=9$. On the other hand, $p^2=EO\cdot OD$; since EO is small compared with OD, we can take $OD=2r$ and we get $9\approx 0.4\cdot 2r$, whence

$$r=\frac{9}{0.8}\approx 11\frac{1}{4}\text{ (m)}$$

155. Radical Axis. Radical Centre

The locus of points M (Figs. 134, 135, 136, 137, 138) having equal powers with reference to two given circles O_1, O_2 ($MK_1=MK_2$) is a straight line AB perpendicular to the line of the centres.

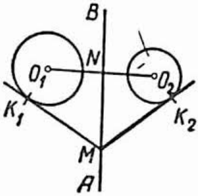


Fig. 134.

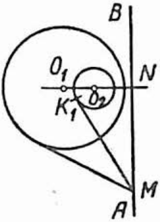


Fig. 135.

This straight line is called the *radical axis* of the circles O_1 and O_2. The distances d_1, d_2 of the radical axis from the centres O_1, O_2 of the given circles may be computed by the formulas

$$d_1 = O_1N = \frac{d}{2} + \frac{r_1^2 - r_2^2}{2d}$$

$$d_2 = NO_2 = \frac{d}{2} + \frac{r_2^2 - r_1^2}{2d}$$

where d is the distance O_1O_2 between the centres of the circles, and r_1 and r_2 are the radii of the circles. It is much easier to find the radical axis by means of a construction. If the circles O_1 and O_2 intersect at points C and D, then each of these points has power zero with respect to both circles and, hence, the radical axis passes through C and D (Fig. 136). If the circles touch at point C (Fig. 137, 138), then the common tangent serves as their radical axis. For nonintersecting circles, the radical axis may be found thus. Construct (Fig. 139) an auxiliary circle O_3 to intersect circle O_1 in points C and D and circle O_2 in points E and F. The straight lines CD, EF are the radical axes of the two pairs of circles O_1, O_3 and O_2, O_3. For this reason, their point P of intersection has the same power with respect to O_1, O_3 as it has with respect to O_2, O_3. Hence, it has the same powers

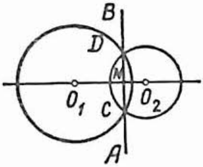


Fig. 136.

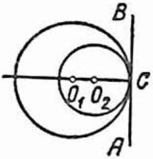


Fig. 137

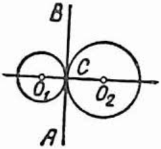


Fig. 138.

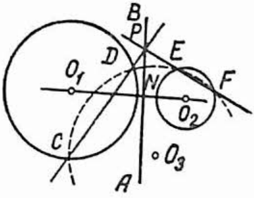


Fig. 139.

with respect to O_1 and O_2, that is to say, it lies on the radical axis of the two given circles. Finding another point in the same manner or dropping a perpendicular PN from P to O_1O_2, we find the required radical axis.

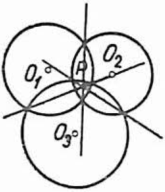


Fig. 140.

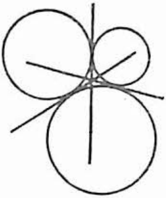


Fig. 141.

The foregoing reasoning shows that the three radical axes of any three circles O_1, O_2, O_3 (taken in pairs) intersect in one point. This point is termed the *radical centre* of the circles O_1, O_2, O_3. In particular, the three common chords of the three pairwise intersecting circles in Fig. 140 intersect in a single point. The three common tangent lines of three pairwise tangent circles (Fig. 141) also intersect in a single point.

156. Inscribed and Circumscribed Polygons

A *polygon inscribed in a circle* is one in which all the vertices lie on the circumference (Fig. 142); a polygon circumscribed about a circle is one in which the sides are tangent to the circle (Fig. 143).

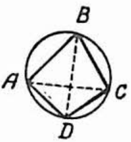


Fig. 142.

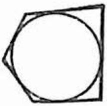


Fig. 143.

A *circle circumscribed about a polygon* is one that passes through the vertices (Fig. 142); a *circle inscribed in a polygon* is one which touches the sides of the polygon (Fig. 143). If we take an arbitrary polygon, a circle can neither be inscribed in it nor circumscribed about it. In the case of a triangle, it is always possible to inscribe a circle in it and circumscribe a circle about it (see Sec. 137 Problems 20, 21 and Sec. 146).

The radius r of an inscribed circle is given in terms of the sides a, b, c of a triangle by the formula

$$r = \sqrt{\frac{(p-a)\,(p-b)\,(p-c)}{p}}$$

where

$$p = \frac{a+b+c}{2}$$

The *radius* R *of a circumscribed circle* is given by the formula

$$R = \frac{abc}{4\sqrt{p\,(p-a)\,(p-b)\,(p-c)}}$$

A circle can be inscribed in a quadrilateral only when the sums of the opposite sides are equal; of all parallelograms, only the rhombus (or square, as a special case) admits inscribing a circle. Its centre lies at the intersection of the diagonals.

A circle can be circumscribed about a quadrilateral only when the sum of the opposite angles is equal to 180° (if this is true for one pair of opposite angles, then the sum of the other pair of angles will definitely come to 180°). Of all parallelograms only the rectangle (square, as a special case) admits circumscribing a circle; its centre lies at the intersection of the diagonals.

A circle can be circumscribed about a trapezoid only when it is an isosceles trapezoid.

In a convex quadrilateral inscribed in a circle, the product of the diagonals is equal to the sum of the products of the opposite sides (Ptolemy's theorem). In Fig. 142

$$AC \cdot BD = AB \cdot DC + AD \cdot BC$$

157. Regular Polygons

A *regular polygon* is a polygon with equal sides and angles. Figure 144 shows a regular hexagon and Fig. 145 a regular octagon. A regular quadrilateral is a square, and a regular triangle is an equilateral triangle. Each angle of a regular n-gon is equal to $\frac{180^\circ\,(n-2)}{n}$.

Referring to Fig. 144 we see that a regular polygon has a point O inside it that is equidistant from all the vertices ($OA = OB = OC$, etc.). This is the centre of the regular polygon. The centre is also equidistant from all the sides of a regular polygon ($OP = OQ = OR$, etc.).

The lines OP, OQ, etc. are called *apothems* (or short radii), and the lines OA, OB, etc., are called the *radii* of a regular polygon.

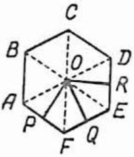


Fig. 144.

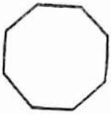


Fig. 145.

The area of a regular polygon is equal to the product of the semiperimeter by an apothem:

$$S = ph$$

where

$$p = \frac{1}{2}(AB + BC + CD + \ldots), \quad h = OP$$

A circle may be inscribed in a regular polygon or circumscribed about it. The centres of the inscribed and circumscribed circles lie at the centre of the regular polygon. The radius of a circumcircle is the radius of the regular polygon; the radius of the inscribed circle is the apothem. (For the construction of inscribed and circumscribed circles of polygons see Sec. 137, Problems 30-38). A side b_n of a regular circumscribed polygon can be expressed in terms of the side a_n of a regular inscribed polygon with the same number of sides by the formula

$$b_n = Ra_n : \sqrt{R^2 - \frac{1}{4}a_n}$$

where R is the radius of the circle.

The side a_{2n} of a regular inscribed polygon with double the number of sides is expressed in terms of a_n by the formula

$$a_{2n} = \sqrt{2R^2 - 2R\sqrt{R^2 - \frac{1}{4}a_n^2}}$$

The following formulas express relationships between the sides of certain regular inscribed polygons and the radius of the circle:

$$a_3 = R\sqrt{3} \approx 1.7321R,$$
$$a_4 = R\sqrt{2} \approx 1.4142R,$$
$$a_5 = R\sqrt{\frac{5-\sqrt{5}}{2}} \approx 1.1755R,$$
$$a_6 = R,$$
$$a_8 = R\sqrt{2-\sqrt{2}} \approx 0.7654R,$$
$$a_{10} = R\frac{\sqrt{5}-1}{2} \approx 0.6180R,$$
$$a_{12} = R\sqrt{2-\sqrt{3}} \approx 0.5176R,$$
$$a_{15} = \frac{1}{4}R\left[\sqrt{10+2\sqrt{5}} - \sqrt{3}\,(\sqrt{5}-1)\right] \approx 0.4158R$$

The expressions for a_3, a_4 and a_6 are frequently used in practical computations; it is best to remember them. The most convenient method for computing the sides of the other polygons is by means of the formulas of trigonometry (see Sec. 190) using tables. For most polygons, the relations $a_n : R$ cannot be expressed as algebraic formulas even if we use nested radical signs (radicals within radicals).

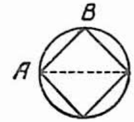


Fig. 146.

Example. Is it possible to obtain a square beam (36 cm on a side) from a log 40 cm in diameter?

We can take the cross section of the log as a circle of radius

$$R = \frac{40}{2} = 20\ (\text{cm})$$

The largest square in this circle is the square that is inscribed in it. Its side AB (Fig. 146) is equal to $20\sqrt{2} \approx 20 \cdot 1.41 \approx 28$ (cm). Hence, we cannot obtain a 36×36 cm beam from this log.

158. Areas of Plane Figures

In this section we give the most important formulas for finding the areas S of plane figures (some of them have already been given in appropriate sections).

Square (Fig. 103, page 274): a is a side, d, a diagonal:

$$S=a^2=\frac{d^2}{2}$$

Rectangle (see Fig. 101, page 274): a and b are sides:

$$S=ab$$

Rhombus (see Fig. 102, page 274): a is a side, d_1 and d_2 are diagonals, and α is one of the angles (acute or obtuse):

$$S=\frac{d_1d_2}{2}=a^2\sin\alpha$$

Parallelogram (see Fig. 100 page 274): a and b are sides, α is one of the angles (acute or obtuse) and h is the altitude:

$$S=ah=ab\sin\alpha$$

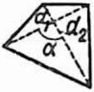


Fig. 147.

Trapezoid (see Figs. 104, 106, page 275): a and b are bases, h, altitude, and c, the median:

$$S=\frac{a+b}{2}h=ch$$

Any quadrilateral: d_1, d_2, diagonals, α, the angle between them (Fig. 147):

$$S=\frac{1}{2}d_1d_2\sin\alpha$$

A *circumscribable quadrilateral* (Sec. 137, Problem 22) a, b, c, d are the sides:

$$p=\frac{a+b+c+d}{2}$$

$$S=\sqrt{(p-a)(p-b)(p-c)(p-d)}$$

Right triangle (see Fig. 75, page 266): a and b are the legs:

$$S=\frac{1}{2}ab$$

An *isosceles triangle* (see Fig. 77, page 267): a, the base, b, a side:

$$S=\frac{1}{2}a\sqrt{b^2-\frac{a^2}{4}}$$

An *equilateral triangle* (see Fig. 78, page 267): a, a side:

$$S=\frac{1}{4}a^2\sqrt{3}$$

An *arbitrary triangle* with a, b, c the sides, a the base, h the altitude, and A, B, C the angles opposite the sides a, b, c, respectively; $p=\frac{a+b+c}{2}$ (Fig. 148):

$$S=\frac{1}{2}ah=\frac{1}{2}ab\sin C=\frac{a^2\sin B\sin C}{2\sin A}=\frac{h^2\cdot\sin A}{2\sin B\sin C}$$
$$=\sqrt{p(p-a)(p-b)(p-c)}$$

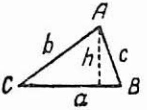


Fig. 148

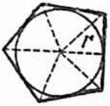


Fig. 149.

A *polygon* whose area is sought is partitioned into triangles in any manner (say by diagonals). It is convenient to partition a polygon, circumscribed about a circle, by straight lines from the centre of the circle to the vertices of the polygon (Fig. 149). We then get

$$S=rp$$

where r is the radius of the circle and p is the semiperimeter.

In particular, this formula is valid for any regular polygon.

A *regular hexagon* in which a is a side:

$$S=\frac{3}{2}\sqrt{3}\,a^2$$

A *circle* in which d is the diameter, r the radius, C the length of the circumference:

$$S=\frac{1}{2}Cr=\pi r^2\,(\approx 3.142r^2)=\pi\frac{d^2}{4}\,(\approx 0.785d^2)$$

A *sector* of radius r; n is the degree measure of the central angle and p_{n° is the arc length (Fig. 150):

$$S=\frac{1}{2}rp_{n^\circ}=\frac{\pi r^2 n}{360}$$

An *annulus* (Fig. 151) where R and r are the outer and inner radii, respectively; D and d, the outer and inner diameters, respectively; $\overline{r}$ the mean radius; and k the width of the annulus:

$$S=\pi(R^2-r^2)=\frac{\pi}{4}(D^2-d^2)=2\pi\bar{r}k$$

Segment. The area of a segment (Fig. 152) is found as the difference between the area of the sector $OAmB$ and the triangle AOB.

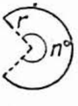


Fig 150.

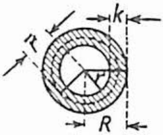


Fig. 151.

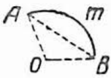


Fig 152.

158a. Approximate Formula for the Area of a Segment

In practical situations, one often has to find the area of a segment given natural-size or in a drawing when it is not known what part of the circle the arc of the segment constitutes and what its radius is. In such cases the following approximate formula may be used:

$$S\approx\frac{2}{3}ah$$

where $a=AB$ (Fig. 152a) is the base of the segment, $h=CM$ is its altitude. In other words, it is taken that the area of a segment is equal to $\frac{2}{3}$ the rectangle $ADEB$. Actually, the

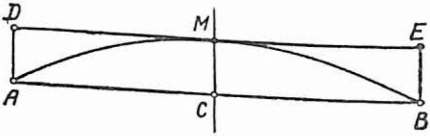


Fig. 152a.

area of the segment is somewhat greater. For $\overset{\frown}{AB}=60°$ the relative error of the formula comes to 1.5%, for $\overset{\frown}{AB}=45°$, it is half as much, for $\overset{\frown}{AB}=30°$ it is 0.3% and continues to fall off even more rapidly.

Example. Find the area of the segment AMB (Fig. 152a) whose base $a=60.0$ mm and altitude $h=8.04$ mm.

Solution. $S \approx \frac{2}{3} \cdot 60.0 \cdot 8.04 \approx 321$ (mm^2). However, the third digit is definitely incorrect since the arc $\breve{AB}$ contains 60° (this is visibly so) and, hence, the error of the formula comes to 1.5%, which is roughly 5 square millimetres. Making the appropriate correction, we find $S \approx 326$ mm^2; all digits are true.

B. SOLID GEOMETRY

159. General

Solid geometry studies the geometric properties of objects and figures in space. When solving problems in solid geometry, an important technique is the consideration of plane lines and figures (both those in the solids under study and also lines and figures constructed as auxiliary elements). It is therefore very important to learn to recognize and isolate diverse plane figures in spatial images.

160. Basic Concepts

As in plane geometry the most elementary line is the straight line, so in solid geometry the most elementary surface is the plane surface — the *plane*. *Planes* and *straight lines* are the basic elements of solid geometry.

One and only one plane can be drawn through any three points of space not all on one straight line. Through three points of one straight line, it is possible to pass an infinitude of planes, called a *pencil* of planes; the line through which all the planes pass is called the *axis* of the pencil.

One and only one plane can be drawn through any straight line and a point external to the line. It is not always possible to draw a plane through two straight lines. Two straight lines through which it is impossible to draw a plane are termed *skew lines*.

Example. A horizontal line drawn on one wall of a room and a vertical line drawn on the opposite wall are skew lines.

Skew lines are nonintersecting lines no matter how far produced, but they are not parallel.

Parallel lines are nonintersecting lines such that a plane can be passed through them (cf. Sec. 148).